

Why We Need Logarithms: From Early Navigation to Nuclear Meltdown

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Introduction

With the implementation of Common Core State Standards for Mathematics (CCCS-M), and the associated assessments, high school students will need frequent problem-solving experience to increase their confidence to persevere through them. I am writing this curriculum unit for high school math students to give them a historical background of logarithms with the hope that they will better understand their usefulness and the justification for their properties. I will pose problems with the ultimate goal of students truly understanding that logs are exponents of a specific base. Once students internalize the idea that logarithms are exponents, I think they will be better prepared to connect the properties of logarithms to the laws of exponents that they have already studied. In this curriculum unit, students will also solve exponential and logarithmic equations. They will apply Polya's four steps for problem solving to answer real questions that can be solved most efficiently with the use of logarithms. The questions relate to nuclear power plant disasters, drug testing, carbon dating, and determining time-of-death based on the cooling rate of a corpse, as seen on crime scene television.

Rationale

I teach in a vocational school district consisting of four high schools that draw from middle schools in all five districts in New Castle County, Delaware. We use the Core Plus integrated math curriculum in 90-minute block periods for one 18-week semester; students have three to four courses per semester. With some exceptions, students take two semesters of Core Plus (1 & 2) in their freshman year, Core Plus 3 in their sophomore year, and beginning in the 2013-2014 school year, either go directly to Precalculus or take a Trigonometry transition course. Students in Precalculus are typically college bound, and the strongest math students in the school. Previously, my Precalculus students had never seen logarithms before, and it is the topic that caused them the most difficulty. However, our district's Core Plus 1, 2, & 3 courses were recently realigned, so that beginning in Fall 2013, students will be introduced to base-10 logarithms in the earlier courses. Therefore, in the future, this curriculum unit may become more appropriate for another math course, when logarithms are first introduced.

My research on the history and use of logarithms brought back my own high school memories. I suddenly realized why the topic is not as intuitive for my students as it was for me. At the risk of dating myself, I went through high school before inexpensive

calculators were readily available. I even had to learn how to use a slide rule, but never actually used one outside of math class. There were base-10 log tables printed on the inside covers of textbooks and we used them to simplify calculations with large numbers. For example, it was much easier to rewrite the division 12,547/942 as subtraction of logarithms (exponents of base 10 read from a table) than to perform long division. After subtracting, we would find the *difference* in the table and convert it back to the *quotient* of the original problem. As I read more history, I realized that today's students have never had to perform tedious long division or root problems because they all have calculators and grew up using them. In fact, some of my students tell me they never learned long division. (I believe they never learned it, but I don't believe they were never taught it!) I'm not suggesting we take away calculators to perform numerical calculations, but I think students will have a greater understanding of what logarithms actually are if they get a sampling of their historical use.

Current math textbooks introduce logarithms with a definition: $y = \log_b x$ if and only if $x = b^y$. The form of the definition may vary, but the basic idea is to define the logarithm as the exponent on base, b , that gives the value x . Students are expected to accept the definition and then the properties of logarithms. From my experience, they generally accept the definition and understand that logarithms are the inverse of exponential functions; logs "undo" exponentials, and can be used to solve for a variable that is an exponent. However, I feel as though they learn the operations and processes by doing rote practice problems, but don't have a solid conceptual understanding to rely on when faced with exponential or logarithmic applications. My goal for this unit is to enhance my students' practice of basic logarithm skills by posing authentic problems that can be modeled by exponential functions, and solved using logarithms and problem-solving strategies. Real-life topics that can be modeled with exponential or logarithmic functions include the decrease of radioactive substances over time, population growth or decay, decrease of medication in the bloodstream over time, cooling of a hot beverage or a dead body, carbon dating – using the level of carbon-14 remaining in an object to determine its age, interest on investments, depreciation of car values, the Richter scale - comparing the strength of earthquakes, sound intensity, acidity or alkalinity measured by the pH scale, and linearization of data, as in statistics

Background

History

This curriculum unit will begin by connecting the historical significance of logarithms to their role in mathematics today. In the first chapter entitled "The History of MATHEMATICS as a Teaching Tool," in an NCTM book, *Historical Topics for the Mathematics Classroom*, Phillip Jones stated "If a perception of structure in an old system makes a segment of mathematics easier to comprehend and extend...then these new perceptions of structure and symbolism make the teaching, learning, and using of

mathematics easier.”ⁱ By taking my students back to a time without calculators and to operations involving large numbers, as in navigation or astronomy, I hope they will appreciate the usefulness of logarithms. According to Frank Swetz in the book *Learn from the Masters*, “...the history of mathematics can supply a structure of understanding relating reasons with results. History can provide a logic between the definition of a mathematical concept and its application....”ⁱⁱ Historically, logarithms converted multiplication into addition, division into subtraction, and roots into division by integers. Addition and subtraction are far simpler operations than multiplication and division, especially as the size of the numbers increases.

John Napier (1550–1617), a Scottish landholder, is generally given the credit as being the inventor of logarithms. He published the book *Mirifici logarithmorum canonis descriptio* in 1614. His book contained tables of logarithms, which reduced the time and increased “the accuracy of carrying out numerical computations of multiplication, division and...roots.”ⁱⁱⁱ Napier invented logarithms to make astronomers’ calculations less time-consuming. A quote I read in multiple sources stated, “the invention of logarithms, by shortening the labors, doubled the life of the astronomer.”^{iv}

Napier’s definition of logarithms was different from our current definition based on exponential notation; exponential notation did not even exist at the time. Napier investigated ratios of sines of angles that were necessary for computations in astronomy. Sines, however, were also defined differently from how they are today. Sines were a specific chord length that depended on the radius of a circle.^v There may be a connection between Napier’s work and the trigonometric identity

$$\sin A \cdot \sin B = \frac{1}{2} \cos(A-B) - \frac{1}{2} \cos(A+B)$$

which also replaces multiplication with addition and subtraction. Ultimately, Napier’s computations were based on the ratio of two velocities, one on a line segment and the other on a ray. It was a ratio of an arithmetic progression and a geometric progression. The word logarithm comes from the Greek words *arithmos*, meaning number, and the *logos*, meaning ratio.^{vi} Napier spent years computing thousands of ratios and then published them for public use, which saved others from having to repeat the calculations each time they needed them. It turned out that Napier’s computations produced what we now refer to as natural logarithms (*ln* of base *e*), although the idea of a base was never part of Napier’s calculations.^{vii}

A London professor of geometry named Henry Briggs (1561-1630) was so inspired by Napier’s work that he traveled to Scotland to consult with Napier. As a result of his visit, Briggs published a different set of logarithm tables in 1624. Briggs’ computations produced what we refer to as common logarithms (log of base 10). Briggs, with agreement from Napier, essentially shifted the scale that Napier used so that the

logarithm of 1 is 0 ($\log 1 = 0$) and the logarithm of 10 is 1 ($\log 10 = 1$).^{viii} This shift made the computations simpler, although Briggs carried out his computations to fourteen places (not so simple!).

Early descriptions of logarithms were different from current ones. For example, Napier's logarithms were based on a radius of 10^7 (used in his calculation of sine ratios). He chose such a large number to ensure that all logarithms were positive integers. Briggs' calculations did not consider logarithms of negative numbers, either. It took until 1637 for symbols for powers of numbers to be introduced by Rene Descartes.^{ix} It was William Oughtred that published the theorems $\log a + \log b = \log ab$, $\log a - \log b = a/b$, and $\log a^m = m \log a$ using words, not symbols, in a booklet in 1652.^x Graphic representations of logarithms were introduced in the 17th century, along with an understanding of functions.^{xi} Finally, in the late 1700's, Leonhard Euler connected exponential and logarithmic functions into a form that we use today, $\log_a b = c$ if and only if $a^c = b$. Or, written as functions, $y = a^x$ and $x = \log_a y$.^{xii}

Theory of Logarithms

Computations Before Calculators

The connection between an arithmetic and geometric progression that the early mathematicians used is illustrated in Table 1. An arithmetic progression progresses by adding a constant value to each term; alternately, an arithmetic progression has a common difference between consecutive terms (theoretically, the difference may be negative or positive). A geometric progression progresses by multiplying each term by a constant value; alternatively a geometric progression has a common ratio between consecutive terms (the ratio may be greater than or less than one, but always greater than zero). This common ratio is the base of the exponential form illustrated in the third row of the table.

0	1	2	3	4	5	6	(arithmetic progression)
1	2	4	8	16	32	64	(geometric progression)
2^0	2^1	2^2	2^3	2^4	2^5	2^6	(geometric, alternate form)

Table 1

Referring to Table 1, the product, $4 \times 8 = 32$, in the geometric progression corresponds to the sum, $2 + 3 = 5$ in the arithmetic progression; the results are aligned in the table. Recognizing that the numbers in the arithmetic progression are the same as the exponents of base-2 corresponding to the values in the geometric progression in the middle row, we can add the exponents (aka logarithms) and use the chart to find the product of two numbers, provided the exponents refer to the same base.

The example given in the previous paragraph along with the exponent rule $a^n \cdot a^m = a^{n+m}$ leads to the first property of logarithms:

$$[1] \quad \log_a(x \cdot y) = \log_a x + \log_a y$$

where the values x and y are any two terms of a geometric progression and $a > 0$ is the base, or common ratio between consecutive terms in the progression. To demonstrate its usefulness in simplifying computations of large numbers, let's find the product $(526,000)(7,890,000)$. We will use base 10 ($a = 10$) so we can refer to a common log table. First, we can write 526,000 as $(5.26 \cdot 10^5)$. I am assuming here that high school students are familiar with scientific notation from science and previous math classes. Writing the number in scientific notation tells us that its logarithm will have a fractional(decimal value between 5 and 6 since $10^5 = 100,000$ and $10^6 = 1,000,000$ and we are looking for a value in between. Next, using the logarithm table in Appendix 3, we find the row for '5.2', and look across to the column labeled '6,' which refers to the value 0.06, to get 0.7210. Then, applying property [1] we find that $\log_{10}(526,000) = 5 + 0.7210 = 5.7210$. The integer piece of the logarithm is called the *characteristic*; it gives us the order of magnitude of the number. The decimal part of the logarithm is called the *mantissa*. Using the same process, we find that $7,890,000 = (7.89 \cdot 10^6)$ and $\log_{10}(7,890,000) = 6.8971$. Applying property [1] again, we get $\log_{10}(526,000 \cdot 7,890,000) = 5.7210 + 6.8971 = 12.6181$. This result tells us that the product of the two large numbers is equal to $10^{12.6181}$. Furthermore, we know that this number is in the trillions because its *characteristic* is 12, which is equal to 10^{12} . To find an equivalent value, we again go to the logarithm table, but this time look "inside" the table (not at row or column headings) for 0.6181. It's in the row labeled "4.1" and the column labeled "5" so $0.6181 = \log(4.15)$, or $10^{0.618} = 4.15$. Finally, we can say that $10^{12.618} = 4.15 \cdot 10^{12} = 4,150,000,000,000$! While it took a lot of words to explain the process, it is still much easier, and possibly more accurate, than multiplying large numbers. Something to keep in mind is that both the table and numbers I used have only 3 or 4 nonzero digits. Multiplication would be even more tedious if there were more nonzero digits.

The second property of logarithms allows us to rewrite division as subtraction. It follows directly from the exponent rule, $\frac{b^n}{b^k} = b^{n-k}$:

$$[2] \quad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

where x and y are, again, any two terms of a geometric progression and $a > 0$ is the base, or common ratio between consecutive terms in the progression. Let's use the example I gave in the Rationale section, $12,547/942$, to demonstrate how to avoid long division.

First, we can write $12,547 = 1.2547 \cdot 10^4$. At this point we have more than 3 nonzero digits, so we have a decision to make – can we round the number to $1.25 \cdot 10^4$, or do we need to interpolate between the values in the table? With the availability of calculators, there is no longer a need to interpolate, so I will not demonstrate the process. So, continuing on, the logarithm table and property [1] gives us $\log_{10}(1.25 \cdot 10^4) = 4.0969$ and $\log_{10}(9.42 \cdot 10^2) = 2.9741$. Finally, applying property [2], $\log_{10}\left(\frac{12,500}{942}\right) = 4.0969 - 2.9741 = 1.1228$. Interpreting this result, we find that the quotient of $12,500/942 = 10^{1.1228}$ which is a value is between $10^1 = 10$ and $10^2 = 100$. Going back to the logarithm table, we find that 0.1228 is the logarithm of approximately 1.33, and therefore, the quotient is $1.33 \cdot 10 = 13.3$.

The third property of logarithms allows us to rewrite powers and roots as multiplication or division by integers. Briggs used this property to find square roots of numbers in creating his logarithm tables. It follows from the exponent rule $(b^k)^n = b^{kn}$:

$$[3] \quad \log_b a^n = n \log_b a$$

First we need to recall that finding the n^{th} root of a number is equal to raising it to the $\frac{1}{n}$ power $(\sqrt[n]{b} = b^{\frac{1}{n}})$. Then, to find the square root of 64, applying property [3] gives us $\log_2(64^{\frac{1}{2}}) = \frac{1}{2} \log_2(64)$. If we refer to Table 1, we find that $\log_2(64) = 6$. Since $\frac{1}{2}(6) = 3$, we have $\log_2(64^{\frac{1}{2}}) = 3$ and $(64^{\frac{1}{2}}) = 2^3 = 8$.

As another example using the base-10 log table in Appendix 3, let's find $\sqrt{5} = 5^{\frac{1}{2}}$. First we will use the log table to find $\log_{10} 5 = 0.6990$. Applying property [3], we get $\log_{10} 5^{\frac{1}{2}} = \frac{1}{2} \log_{10} 5 = \frac{1}{2}(0.6990) = 0.3495$. Going back to the log table, we find $\sqrt{5} \approx 2.24$.

Before moving on, I would take some time with students to examine the logarithm table a little more closely. I think it might be interesting, or at least worthwhile for them to see that the values in the table do not increase at a constant rate throughout the table. There are greater changes in the mantissa for smaller numbers than there are for larger numbers. If time allows, students may be interested to see logarithmic graph paper and experiment with graphing some exponential or power (x^2 or x^3) relations.

Inverse Functions – Still a Necessity with Calculators

While calculators have eliminated the need for logarithms as a way to simplify computations, logarithms are still needed for math and science applications. Specifically, the common log is the inverse of the exponential function, $f(x) = 10^x$ and the natural log is the inverse of the exponential function, $f(x) = e^x$. My Precalculus students have studied exponential functions in prior math courses. They are able to recognize growth and decay functions from patterns in a graph and/or table, and they are able to write the function rule based on data or information given. However, before Precalculus, the only way they know to solve an exponential equation is by searching for values in a table or on a graph using a graphing calculator (or by guess & check). Logarithms are, therefore, a valuable mathematical concept if only to reinforce inverse operations in solving equations. As stated in the CCCS-M Function Domain (HSF-BF.B.4), students should be able to “find inverse functions.” They should be able to solve equations using inverses, write the inverse, verify that a function is the inverse of another, and know when a function has an inverse and, when it doesn’t, how to produce an inverse by restricting the domain. Standard HSF-BF.B.5 states that students should “understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.”

Typical, Traditional, Textbook Logarithm Lessons

As stated earlier, most textbooks present a definition of logarithms such as

$$x = \log_b y \text{ if and only if } y = b^x,$$

followed by the three properties of logarithms

$$\begin{aligned} [1] \quad & \log_a(x \cdot y) = \log_a x + \log_a y \\ [2] \quad & \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y \\ [3] \quad & \log_b a^n = n \log_b a. \end{aligned}$$

Students practice using the definition and properties to rewrite expressions such as

- a) $3^n = 243 \rightarrow \log_3 243 = n$
- b) $\log_7(8x) = 5 \rightarrow 7^5 = 8x$
- c) $\log_2\left(\frac{5b^3}{dh^4}\right) \rightarrow \log_2 5 + 3\log_2 b - \log_2 d - 4\log_2 h$
- d) $\log_{\sqrt[3]{3}}(7) + \log_{\sqrt[3]{3}}n + \frac{1}{3}\log_{\sqrt[3]{3}}a - \log_{\sqrt[3]{3}}b \rightarrow \log_{\sqrt[3]{3}}\left(\frac{7na^{\frac{1}{3}}}{b}\right)$

Students typically practice enough to apply the definition and properties by rote. I do not intend to take this practice out of my curriculum, but I do intend to enhance it with problems and/or applications that demonstrate the need for these skills.

Once students have mastered the skills to work with exponential and logarithmic expressions, they begin to solve equations. The simplest exponential and logarithmic equations to solve are based on One-to-One Properties: $a^x = a^y$ if and only if $x = y$, and $\log_a x = \log_a y$ if and only if $x = y$.

$$\begin{aligned} \text{Example A: } & 2^{x+3} = 32 \\ & 2^{x+3} = 2^5 \\ & x + 3 = 5 \\ & x = 2 \end{aligned}$$

*rewrite 32 using base 2
One-to-One Property
subtract 3 from each side*

$$\begin{aligned} \text{Example B: } & \log_4(2x-5) = \log_4(x+4) \\ & 2x - 5 = x + 4 \\ & x = 9 \end{aligned}$$

*One-to-One Property
add 5, subtract x from both sides*

To solve an exponential equation with only one exponential term, the first step is to apply inverse arithmetic operations (addition/subtraction and multiplication/division), if necessary, to isolate the exponential term with the variable in its exponent. Once isolated, we apply the inverse of an exponential, which is a logarithm, keeping the equation balanced by performing the same operation on both sides. If necessary, we would apply additional inverse operations to isolate the variable. The following example could represent the time it takes to double a population that grows at a rate of 4% per year.

$$\begin{aligned} \text{Example: } & 250(1.04)^x = 500 \\ & (1.04)^x = 2 \\ & \log (1.04)^x = \log 2 \\ & x \cdot \log 1.04 = \log 2 \\ & x = \frac{\log 2}{\log 1.04} = 17.67 \end{aligned}$$

*divide both sides by 250
take log (or ln) of both sides
property [3]
divide by log 1.04; enter in calculator*

To solve an equation with a single logarithm, we will again need to isolate the logarithm first, and then apply the inverse of a logarithm, which is exponential. (The process of changing from logarithm to exponential is called *exponentiating*.)^{xiii} Then, if necessary, apply additional inverse arithmetic operations to isolate the variable. The following example from chemistry would be used to find the concentration of the hydronium ion, $[H^+]$, in a solution that has $pH = 7.5$.

$$\begin{aligned} \text{Example: } & -\log_{10}(x) = 7.5 \\ & \log_{10}(x) = -7.5 \end{aligned}$$

*definition of pH=7.5
divide both sides by -1*

$$x = 10^{-75}$$

*exponentiate with base 10
No additional steps required.*

Some logarithm equations require us to use the properties of logarithms to combine terms into a single logarithm before we can apply the inverse operation. The following example does not have a context, but illustrates skills used to solve log equations, provided the bases for all of the logarithms are the same.

Example: $\log_6(3x+14) - \log_6 5 = \log_6(2x)$

$$\log_6\left(\frac{3x+14}{5}\right) = \log_6(2x) \quad \text{property [2]}$$
$$\frac{3x+14}{5} = 2x \quad \text{One-to-One Property}$$
$$3x + 14 = 10x \quad \text{multiply both sides by 5}$$
$$14 = 7x \quad \text{subtract } 3x \text{ from both sides}$$
$$x = 2 \quad \text{divide both sides by 7}$$

Before moving on, I want to add a disclaimer. The CCCS-M standards expect students to work with base 2, 10 and e . This unit does not specifically address base e because of the space required to explain its origin. However, all properties and operations that I have discussed also apply to base e . (In reality, I personally choose natural logs most naturally when solving exponential equations.)

Problem-Solving Strategies

In our seminar, we solved a wide variety of problems. We always compared and discussed all of the different solutions methods we used. We discussed the mathematics involved in each of the problems, and ways to adapt them for varying grade levels. We emphasized problem-solving strategies from G. Polya's book, first published in 1945, *How to Solve It.*^{xiv} Polya outlined and described these four basic steps:

1. Understanding the Problem
2. Devising a Plan
3. Carrying Out the Plan
4. Looking Back

Within each of Polya's four steps, there are questions we want to encourage our students to ask themselves. The questions Polya suggested in 1945 sound just like the first CCSS Standard for Mathematical Practice (CCSS.Math.Practice.MP1): "Make sense of problems and persevere in solving them."

Step 1: Understanding the Problem

When faced with a problem, Polya and CCSS-M suggest asking, “What is the unknown?” “What information is given?” “What information is not given?” “What do we know about the conditions/constraints?” “What prior knowledge do we have that can help us to get started on the problem?” “Can we draw a picture or separate the problem into parts to help us get started?”

Step 2: Devising a Plan

Some questions we want students to ask themselves when approaching a new problem include, “Have we seen any similar problems before?” “What is/are the connection(s) between the givens and what I need to find (the unknown)?” “Is there a simpler form or special case of this problem we can solve that might lead us toward a solution of the original problem?” “What tools (theorems, properties, definitions, etc.) should we use to solve the problem?” “Can we restate the problem differently?” “Can we find something useful from the given data?” “What other data might we need before we can solve for the unknown?”

Step 3: Carrying Out the Plan

In this step, we need our students to persevere and continually ask, “Does this step make sense?” “Are we sure it is correct?” If students find they are not getting closer to a solution, we want them to change their plan, and carry it out.

Step 4: Looking Back

Again, students should ask themselves, “Does this solution make sense?” Other questions include “Are there other methods we could have used?” “Is one method more efficient than another?” “Can we use this result for some other problem(s)?” “What strategies worked?” “What strategies didn’t work?”

Because all of the problems I will pose to my students are related to one general topic, my emphasis will be on the first two steps – identifying the information given, the information needed and how to organize it. That doesn’t mean I will ignore the last two steps, especially since these steps require students to monitor themselves and check whether each step in their plan and their solution make sense. However, in carrying out their plans, I expect them to practice their skills in working with logarithms.

Activities

Lesson 1 – History of Logarithms

Warm-up: Review rules of exponents.

Students will practice simplifying expressions such as $(3^2 a^4 \cdot 3^2 a^3 \cdot a) \left(\frac{6^2 b^8 c^2}{6^3 b^4 c^5} \right)$, and $(5x^4)^3$. If necessary, I will have them write each individual factor to reinforce the rules. For example, $(3^2 a^4 \cdot 3^2 a^3 \cdot a) = 3 \cdot 3 \cdot aaaa \cdot 3 \cdot 3 \cdot aaa \cdot a = 3^4 a^8$ and $(5x^4)^3 = (5xxxx)(5xxxx)(5xxxx) = 5^3 x^{12}$. Having students write the intermediate step for $(5x^4)^3$ will help them remember that both the number 5 and x^4 are being multiplied 3 times (a common error is forgetting to raise the constant, 5, to the power of 3).

Instruction:

Students will complete tables, similar to Table 1. The first row of each table contains integers beginning with zero, increasing by one. The second row contains the values of any base number multiplied by itself (raised to the power of) the number of times corresponding to the integer in row one. The third row contains the values from row two written in exponential form. Students will complete two or three tables using different base values. The third table will have a base value between zero and one, the reciprocal of one of the previous tables; values in row 2 should be written as fractions, rather than decimals.

Base/multiplier = 10

0	1	2	3	4	5	6	7	8
1	10	100	1000	10,000	100,000	1,000,000	10,000,000	100,000,000
10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8

Table 2

Base/multiplier = $\frac{1}{10}$

0	1	2	3	4	5	6	7	8
1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10000}$	$\frac{1}{100000}$	$\frac{1}{1000000}$	$\frac{1}{10000000}$	$\frac{1}{100000000}$
$\left(\frac{1}{10}\right)^0$	$\left(\frac{1}{10}\right)^1$	$\left(\frac{1}{10}\right)^2$	$\left(\frac{1}{10}\right)^3$	$\left(\frac{1}{10}\right)^4$	$\left(\frac{1}{10}\right)^5$	$\left(\frac{1}{10}\right)^6$	$\left(\frac{1}{10}\right)^7$	$\left(\frac{1}{10}\right)^8$

Table 3

Referring to row one in Tables 2 and 3, define an **arithmetic progression/sequence** as an ordered list of numbers in which any two consecutive terms have a common

difference. The common difference in both tables is 1. Referring to row two in both tables, define a **geometric progression/sequence** as an ordered list of numbers in which any two consecutive terms have a common ratio. The common ratio is the multiplier identified at the top of Tables 2 and 3. Challenge students to recognize the pattern in row one as linear, and the pattern in row two as exponential.

Next, ask students to find the product of any two numbers from row two (without a calculator!), and then find that product in the table (extend the table, if necessary). Compare the relationship with the corresponding values in row 1. For example, suppose students choose $\frac{1}{100} \cdot \frac{1}{10000} = \frac{1}{1000000}$ from Table 3. These values correspond to 2, 4 and 6 in row one. Students may notice that $2 + 4 = 6$. They should try several more products until they see the relationship. They should repeat the process for other tables. If students are working in pairs, or groups, they should compare their findings with several different base/multiplier values. Students should discover that the sum of the numbers in the arithmetic sequence corresponds to the product of the corresponding numbers in the geometric sequence, and that the relationship holds for all base values.

At this point, describe how a Scotsman, John Napier, recognized this relationship in the late 1500's. Over more than a decade, Napier calculated thousands of ratios connecting arithmetic and geometric progressions, called logarithms, and published them in a book in 1614. Another mathematician in Britain, named Henry Briggs, was so interested in Napier's logarithms that he traveled to Scotland to discuss them with Napier. They agreed on some adjustments, and Briggs published another set of logarithms in 1624 using different calculations based on (square) roots of numbers. Having these calculations published saved navigators and astronomers from having to repeat calculations whenever they needed them.

To continue illustrating relationships in Tables 2 and 3, again ask students to select two numbers from the second row, and divide them. I suspect they will quickly catch on that the difference of the numbers in the arithmetic sequence corresponds to the quotient of numbers in the geometric sequence. The third relationship students should observe is connected to the rule for exponents: $(a^m)^n = a^{mn}$, or a number raised to a power (exponent), raised to a second power is equal to the number raised to the product of the two powers. Using any number in the geometric sequence of a table, students should square the number, cube the number and find the square root of that number to find the relationship with the corresponding values in the arithmetic sequence. For example, using the number 1000 in row two of Table 2, $1000^2 = 1,000,000$ and the corresponding values in row one are 3 and 6. Likewise, $1000^3 = 1,000,000,000$, and by extending the table one column, the corresponding values in row one, are 3 and 9. To find square roots, they may need to be more strategic by selecting only even values in the arithmetic sequence. For example, starting with the number 4 in the arithmetic sequence, the square root of the corresponding value, 10,000, is 100, corresponding to 2 in row one. By comparing results

with other students, they should discover that squaring and cubing a number in the geometric sequence corresponds to multiplying the number in the arithmetic sequence by 2 and 3, respectively. Square root corresponds to dividing the exponent by 2, which is the same as multiplying by $\frac{1}{2}$.

In the final piece of this lesson, ask students if they have used the three relationships they just observed in any other contexts in mathematics. If no one can verbalize the relationships as the rules of exponents, draw their attention to the third row of each table. Finally, *define* the values in row one as the **logarithms** of the base. Referring to the third row in the table, students should recognize that the logarithm is the exponent on the base, and therefore, rules of exponents apply to logarithms.

Formative Assessment (sample):

Since $10^4 = 10,000$, we can write $\log_{10}(10,000) = 4$. Since $2^4 = 16$, we write $\log_2(16) = 4$.

Write the following statements as logarithms: $5^3 = 125$, $12^2 = 144$, and $\left(\frac{2}{5}\right)^{-1} = \frac{5}{2}$.

Activity 2 – Base 10 Logarithms

Warm-up: Apply number sense to estimate numbers using exponents of base 10.

Students should estimate the exponent on base 10 between two consecutive integers for several positive integers, both greater than and less than one. A sample question set with answers may look like:

- | | |
|---------------------|---|
| 1) $355 = 10^x$ | Since $100 \leq 355 \leq 1000$, $2 \leq x \leq 3$ |
| 2) $12,498 = 10^x$ | Since $10,000 \leq 12,498 \leq 100,000$, $4 \leq x \leq 5$ |
| 3) $0.00475 = 10^x$ | Since $0.001 \leq 0.00475 \leq 0.01$, $-3 \leq x \leq -2$ |

The next step is to review how to write the numbers in scientific notation as 1) 3.55×10^2 , 2) 1.2498×10^4 , and 3) 4.75×10^{-3} . This form of the numbers may help students that were unsure how to estimate the exponents in the warm-up questions.

Instruction

After a brief discussion about real number exponents existing for numbers such as the ones in the Warm-up questions, ask students to use “Guess and Check” to find a more accurate estimate of the exponent of base 10. Then, it’s time to introduce a Base 10, or Common, Logarithm Table (referred to as “log table” for the remainder of this unit). Using the same set of numbers, demonstrate how to use the log table for numbers $1 \leq x \leq 10$, the first part the number written in scientific notation. This activity reinforces

exponent rules by breaking numbers written in scientific into two factors having the same base, so the exponents can be added:

- 1) $355 = 3.55 \times 10^2 = 10^{0.5502} \times 10^2 = 10^{2.5502}$
- 2) $12,498 = 1.2498 \times 10^4 = 10^{0.0969} \times 10^4 = 10^{4.0969}$
- 3) $0.00475 = 4.75 \times 10^{-3} = 10^{0.6767} \times 10^{-3} = 10^{-2.3233}$

Question #3 above can generate discussion about the relative size of 10^{-2} versus 10^{-3} , and how adding a positive value from the log table to the negative exponent from scientific notation leads to a less negative exponent which is larger than -3. Also, provide practice using the log table to convert between base 10 logarithms and the equivalent number written in standard form. For example, complete Table 4 below.

Base 10 Logarithm	Given: $\log x = 3.5888$	Given: $\log x = 1.8579$	Given: $\log x = -2.1500$	Given: $\log x = -0.7190$
Standard Form	Ans: $x=3880$	Ans: $x=72.1$	Ans: $x=0.00708$	Ans: $x=0.191$

Table 4

The goal of the next phase of this lesson is to demonstrate to students why log tables were useful and timesaving before calculators were available. (They can't imagine life without a calculator, and hearing that textbooks used to have these tables printed on the covers fascinates them.) The following problems make use of the tables and exponent rules. First, students should pick a number from 10–100 and write it in the form 10^x (find its logarithm using a base 10 log table). Second, pick a number from 100–1000 and write it in the form 10^x . The first problem is to multiply the two numbers in their original form by hand. Then, multiply using the exponential form of the numbers by adding the exponents. Demonstrate how to use the log table to transform the logarithm of the product back into a number between 1-10 written in standard form.

The second problem is to perform long division using the same two numbers by hand. Then, perform the same division using the exponential form of the numbers by subtracting the exponents and convert it back into standard form. The third problem is to square and cube one of the numbers by hand, and then using the exponential form of the number. Finally, ask students to estimate the square root of the number between 100–1000, and refine their estimate using Guess and Check. As a final illustration of the power of logarithms, demonstrate that the square root of a number can be found by dividing its logarithm by 2 and transforming the result back to standard form.

Now, students should be prepared to learn and work with the three properties of logarithms described in the Background section above. They can and should practice

rewriting logarithms of products and quotients as the sum and difference of logarithms. They should also practice going in reverse to condense a sum/difference into a single logarithm. The practice problems should include both variables and numbers

Formative Assessment:

As an Exit Ticket, have students summarize in their own words the process of division using logarithms. For example, perform the division 1265/987 while describing the process. (Of course, division can be replaced with multiplication or finding the square, cube or square root of numbers.)

Activity 3 – Logarithms and Exponentials are Inverse Functions

Warm-up: Practice converting between exponential and logarithmic form

Exponential form	Given: $\left(\frac{1}{10}\right)^3 = \frac{1}{1000}$	Ans: $2^6 = 64$	Ans: $\left(\frac{1}{10}\right)^{-3} = 1000$	Given: $\left(\frac{4}{9}\right)^{-2} = \frac{81}{16}$
Logarithmic Form	Ans: $\log_{1/10}\left(\frac{1}{1000}\right) = 3$	Given: $\log_2(64) = 6$	Given: $\log_{1/10}(1000) = -3$	Ans: $\log_{4/9}\left(\frac{81}{16}\right) = -2$

Table 5

Instruction:

Begin instruction by teaching Polya's four steps in problem-solving: 1) Understand the problem, 2) Devise a plan, 3) Carry out the plan, and 4) Look back to be sure the answer makes sense. Have students apply Polya's four steps in solving a sampling of equations algebraically, such as a) linear: $6x - 5 = 13$, b) quadratic: $(2x+1)^2 = 5$, c) power: $4x^3 = 32$, and d) reciprocal: $\frac{1}{x} = \frac{2}{x+5}$. Explicitly discuss how understanding the problem means recognizing the type of equation (linear, quadratic, etc.) and what it means to solve the equation. In devising a plan, students should describe how to use inverse operations to isolate variables in the different types of equations. They should then carry out their plan to find all solutions, and finally, look back to check their solutions.

Next, present students with a problem such as $20(2)^{4x} = 1280$. This problem could describe bacteria growing in a Petri dish for x hours if there were 20 bacteria initially, each doubling every 15 minutes, and we want to know how long it will take to reach 1280 bacteria. Hopefully students will recognize this equation as exponential with the

variable x representing time, given in hours. Using the previous lesson and warm-up activity, I hope students' plan will be to rewrite the exponential equation as a logarithm. Ensure that they isolate the exponential by dividing by 20 (inverse of multiplication) on both sides of the equation, simplifying the original to $(2)^{4x} = 64$. Rewriting in log form gives us $\log_2(64) = 4x$. If students have not yet learned the Change of Base Theorem (mine have not), they will need to recognize that 64 can be written with base two as 2^6 . Then the exponent $4x$ must equal 6, and after dividing again, we find that $x = 6/4 = 1.5$. Ensure that students interpret the result as after 1.5 hours of doubling there will be 1280 bacteria in the dish. To check, substitute $x = 1.5$ and multiply $20 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 1280$.

We could change the bacteria example slightly by describing a different Petri dish that starts with 10 bacteria with each doubling every 5 minutes. When would the two dishes have the same number of bacteria? The equation to solve would be $20(2)^{4x} = 10(2)^{12x}$. The starting numbers make it an interesting problem with multiple approaches possible. Solution #1: Divide both sides by 10 to get $2(2)^{4x} = 2^{12x}$, which becomes $2^{4x+1} = 2^{12x}$. If students recognize that the exponents must be equal, then $4x+1=12x$, $x=1/8$ hr = 7.5 min. Solution #2: Rewrite both sides in log form to get $\log_2(20(2)^x) = \log_2(10(2)^x)$. Using properties of logarithms, $\log_2(20) + 4x\log_2(2) = \log_2(10) + 12x\log_2(2)$. Subtracting $4x\log_2(2)$ and $\log_2(10)$ from both sides gives us $\log_2(20) - \log_2(10) = 12x\log_2(2) - 4x\log_2(2)$. By using log properties on the left and combining like terms on the right, we get $\log_2\left(\frac{20}{10}\right) = \log_2(2) = 8x\log_2(2)$. Dividing both sides by $\log_2(2)$ we end up with $1=8x$, so $x=1/8$ hr. Students may also recognize (I can hope) that $\log_2(2) = 1$ which is another way to simplify the right side of the equation to $12x - 4x = 8x$.

Challenge students to find a solution to a related bacteria problem in which 20 bacteria in one dish doubles every 15 minutes while 10 bacteria in the other triples in the same time period. When would both dishes contain the same number of bacteria? The equation to solve would be $20(2)^{4x} = 10(3)^{4x}$. Students now need to decide what to do with two different bases. Do they rewrite $(2)^{4x}$ using base 2 and $(3)^{4x}$ using base 3? Or do they use the same base for both, and if so, what base? Perhaps they will think to use base 10 for both since they know how to find base 10 logarithms from a table, and possibly their calculator. The follow-up discussion should lead to the conclusion that logarithms and exponentials are inverse functions.

Formative Assessment:

Solve for x .

1) $\log_2(2x - 3) = 3$

Answer: $2^3 = 2x - 3$; $x = 5.5$

2) $\log(2x - 3) + \log(x - 2) = 2\log(x)$

Answer: $(2x-3)(x-2)=x^2$; $x^2 - 7x + 6 = 0$;

$(x-6)(x-1)=0$; $x=6$ ($x=1$ does not satisfy)

$$3) 7 + 2\log(x) = 6$$

$$\text{Answer: } \log(x) = -1/2; \quad x = 10^{-1/2} = \frac{1}{\sqrt{10}}$$

At this point, I would refer to a textbook to practice solving all forms of log or exponential equations, described in the Background section, using inverse operations and inverse functions to isolate and solve for variables. Now that students have a deeper understanding of what logarithms are, it would be appropriate to demonstrate how to use calculators to find base 10 logarithms instead of using the tables. It may also be appropriate to formalize the “Change of Base Theorem,” although they have already used it informally in examples they have seen. Finally, I think it would be appropriate to introduce natural logarithms at this time.

Activity 4 – Real Problems/Project

Warm-up: Before beginning this project, students will need to do some background research, either in class or at home. If desired, it can be a “Jigsaw” activity where students are assigned specific questions that they will explain to others. Some useful websites are listed in Appendix 2. They should read enough to find answers to the following questions:

1. What is radioactivity? What is half-life?
2. What are effects/risks of exposure to radiation?
3. Uranium has several isotopes. What is the most abundant isotope, and what must be done to make it useful for nuclear energy?
4. How does a nuclear power plant work?
5. What are the (most common) by-products of nuclear fission?
6. Radiation releases can contaminate the air, water and soil. What are the EPA maximum exposure levels?

Instruction:

Armed with a little bit of knowledge, it’s time apply Polya’s problem solving steps as we work through an example from the March 11, 2011 Fukushima Power Plant disaster in Japan. One report from *NHK-world* (15 November 2011) found soil contamination of 39,000 becquerels/kg of radioactive cesium-137. The government’s safety limit is 5,000 becquerels/kg, and the half-life of cesium-137 is approximately 30 years.

Step 1: Understand the problem. The problem is “How long will it be before it is safe for people to be outside, exposed to the soil, or to plant food in the soil?” The background reading should help students understand the context and vocabulary in the problem. They should understand that cesium-137 is a radioactive material that decays slowly; after 30 years, half of the initial contamination remains in the soil, and after another 30 years (60

years, total), half of that (one-fourth of the initial) still remains. Students should recognize this pattern as exponential decay, and that we need to solve for t when $y = 5000$.

Step 2: Devise a plan. First, we will need to write an equation that models radioactive decay using the general exponential form students have learned, $y = a(b)^x$, where a is the starting value (also the y -intercept), and b is the base that is multiplied repeatedly. For exponential decay, students know that the b value is less than one. A brief discussion should establish that the variable x represents time (in years) since the contamination was first measured. The information given tells us that $t = 30$ years when one-half of the original amount remains. We can substitute $a = 40,000$ and $y = 20,000$. The remaining unknown, then, is b . I will suggest we use $b = 10$ now that they have learned to solve exponential equations using base 10 logarithms. But, that means we have substituted for all of the variables, so what are we solving for? There must be something unique about the exponential decay equation for each radioactive material. Reminding students that base 10 raised to the power of 30 will increase the amount of cesium-137, I think I can convince them to add a “constant” parameter, k , to the exponent and see what happens.

Step 3: Carry out the plan. Substitute all known values, and solve for k .

$$20,000 = 40,000(10)^{30k}$$

$$\begin{aligned} \frac{1}{2} &= 10^{30k} && \text{(divide by 40,000)} \\ \log \frac{1}{2} &= \log 10^{30k} && \text{(take log of both sides)} \\ \log \frac{1}{2} &= 30k \cdot \log 10 && \text{(apply property [3])} \\ \log \frac{1}{2} &= 30k && \text{(\log 10 = 1)} \\ \frac{\log \frac{1}{2}}{30} &= k \approx -0.0100 && \text{(divide by 30; enter in calculator)} \end{aligned}$$

This result, a negative value for the exponent k , makes sense – a negative exponent means we are actually multiplying by $\frac{1}{10^{0.01}}$, which will give us decreasing values of radiation over time. Therefore, the equation for radioactive decay of cesium-137 is $y = 40,000(10)^{-0.01t}$. We can work with this, or we can apply the power rule of exponents to rewrite it as $y = 40,000(10^{-0.01})^t = 40,000(0.977)^t$. To answer the question about how long until it is safe to be exposed to the soil, we can substitute the safety limit, $y = 5000$, and solve for t .

$$\begin{aligned}
 5,000 &= 40,000(.977)^t \\
 \frac{5,000}{40,000} &= (.977)^t \\
 \log(.125) &= t \cdot \log(.977) \\
 \frac{\log(.125)}{\log(.977)} &= t \approx 89 \text{ years}
 \end{aligned}$$

Step 4: Look Back. The numbers in this example make it very easy to confirm that our process and answer makes sense. If the initial contamination is 40,000 bq/kg with a half-life of 30 years, then 20,000 bq/kg will remain after 30 years, 10,000 after 60 years, and 5,000 after 90 years. Students should be able to explain why our result of 89 years is essentially the same; the difference arises from rounding numbers through the process. When people are ready to resume planting food in the soil, they will test it at that time to ensure it meets safety requirements, which could very easily change during the 90 years!

Final Assessment/Project:

Students may work alone, or in groups of two or three. They will conduct research on a topic of their choice that can be modeled with an exponential or logarithmic function. Each group will formulate a question that requires application of logarithms, and get approval from the teacher before continuing. The final product will be a presentation using computer slides or poster. The key components of the presentation will demonstrate Polya's four problem solving steps: 1) a summary of the problem so observers will understand, 2) the plan(s) considered to help answer the question, including any additional information they may need, 3) all of the steps used in carrying out the plan (including going in the wrong direction and how they knew and changed plans, if that occurred) and 4) a clear interpretation of the answer to the question.

Some possible scenarios for students to explore are

- a) whether a drug test for a job will detect illegal drug use if the test is done one week after a drug was used.
- b) determining the time of death of a corpse based on body temperature similar to what they see on crime scene TV shows using Newton's law of cooling
- c) using carbon-14, a radioactive isotope of carbon, to calculate the age of a skeleton.
Carbon dating uses the fact that living organisms constantly ingest carbon-14. After death, no more is ingested, and what was present at the time of death decreases over time with a half-life of 5,700 years.
- d) radioactive contamination of water, soil, or air from Fukushima, or from other nuclear power plant crises (Chernobyl, Three-Mile Island).

Appendix 1 – Standards

This unit addresses CCSS-M Content standards of Number and Quantity (The Real Number System) and Functions (Building Functions, Linear, Quadratic, & Exponential Models), HSF-LE.B.5,

CCSS.Math.Content.HSN-RN.A.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

CCSS.Math.Content.HSF-BF.B.4 Find inverse functions.

CCSS.Math.Content.HSF-BF.B.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

CCSS.Math.Content.HSF-LE.A.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.

CCSS.Math.Content.HSF-LE.A.2 Construct linear and exponential functions, including arithmetic and geometric sequences, give a graph, a description of a relationship, or two input-output pairs.

CCSS.Math.Content.HSF-LE.A.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a , c , and d are numbers and the base b is 2, 10 or e ; evaluate the logarithm using technology.

CCSS.Math.Content.HSF-LE.B.5 Interpret the parameters in a linear or exponential function in terms of a context.

This unit also addresses most of the Mathematical Practice standards, especially:

CCSS.Math.Practice.MP1 Make sense of problems and persevere in solving them.

CCSS.Math.Practice.MP3 Construct viable arguments and critique the reasoning of others.

CCSS.Math.Practice.MP4 Model with mathematics.

CCSS.Math.Practice.MP5 Use appropriate tools strategically.

Appendix 2 – Nuclear Power and Radiation websites

The websites listed are ones I found useful. There are links to more information from

each of these websites. Students should follow links of interest to learn more about nuclear radiation and nuclear energy.

www.epa.gov/radiation/

www.nrc.gov/reading-rm/doc-collections/fact-sheets/enrichment.html

www.who.int/mediacentre/factsheets/fs371/en/index.html

www.epa.gov/radiation/docs/402-k-10-008.pdf

<http://science.howstuffworks.com/nuclear-power.htm>

Appendix 3 – Logarithm Table

Common Logarithm Table

N	0	1	2	3	4	5	6	7	8	9	N	0	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170	0.0212	0.0253	0.0294	0.0334	0.0374	5.5	0.7404	0.7412	0.7419	0.7427	0.7435	0.7443	0.7451	0.7459	0.7466	0.7474
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755	5.6	0.7482	0.7490	0.7497	0.7505	0.7513	0.7520	0.7528	0.7536	0.7543	0.7551
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106	5.7	0.7559	0.7566	0.7574	0.7582	0.7589	0.7597	0.7604	0.7612	0.7619	0.7627
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430	5.8	0.7634	0.7642	0.7649	0.7657	0.7664	0.7672	0.7679	0.7686	0.7694	0.7701
1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732	5.9	0.7709	0.7716	0.7723	0.7731	0.7738	0.7745	0.7752	0.7760	0.7767	0.7774
1.5	0.1761	0.1790	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014	6.0	0.7782	0.7789	0.7796	0.7803	0.7810	0.7818	0.7825	0.7832	0.7839	0.7846
1.6	0.2041	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279	6.1	0.7853	0.7860	0.7868	0.7875	0.7882	0.7889	0.7896	0.7903	0.7910	0.7917
1.7	0.2304	0.2330	0.2355	0.2380	0.2405	0.2430	0.2455	0.2480	0.2504	0.2529	6.2	0.7924	0.7931	0.7938	0.7945	0.7952	0.7959	0.7966	0.7973	0.7980	0.7987
1.8	0.2553	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765	6.3	0.7993	0.8000	0.8007	0.8014	0.8021	0.8028	0.8035	0.8041	0.8048	0.8055
1.9	0.2788	0.2810	0.2833	0.2856	0.2878	0.2900	0.2923	0.2945	0.2967	0.2989	6.4	0.8062	0.8069	0.8075	0.8082	0.8089	0.8096	0.8102	0.8109	0.8116	0.8122
2.0	0.3010	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.3160	0.3181	0.3201	6.5	0.8129	0.8136	0.8142	0.8149	0.8156	0.8162	0.8169	0.8176	0.8182	0.8189
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404	6.6	0.8195	0.8202	0.8209	0.8215	0.8222	0.8228	0.8235	0.8241	0.8248	0.8254
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.3560	0.3579	0.3598	6.7	0.8261	0.8267	0.8274	0.8280	0.8287	0.8293	0.8299	0.8306	0.8312	0.8319
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784	6.8	0.8325	0.8331	0.8338	0.8344	0.8351	0.8357	0.8363	0.8370	0.8376	0.8382
2.4	0.3802	0.3820	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962	6.9	0.8388	0.8395	0.8401	0.8407	0.8414	0.8420	0.8426	0.8432	0.8439	0.8445
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133	7.0	0.8451	0.8457	0.8463	0.8470	0.8476	0.8482	0.8488	0.8494	0.8500	0.8506
2.6	0.4150	0.4166	0.4183	0.4200	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298	7.1	0.8513	0.8519	0.8525	0.8531	0.8537	0.8543	0.8549	0.8555	0.8561	0.8567
2.7	0.4314	0.4330	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.4440	0.4456	7.2	0.8573	0.8579	0.8585	0.8591	0.8597	0.8603	0.8609	0.8615	0.8621	0.8627
2.8	0.4472	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609	7.3	0.8633	0.8639	0.8645	0.8651	0.8657	0.8663	0.8669	0.8675	0.8681	0.8686
2.9	0.4624	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757	7.4	0.8692	0.8698	0.8704	0.8710	0.8716	0.8722	0.8727	0.8733	0.8739	0.8745
3.0	0.4771	0.4786	0.4800	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.4900	7.5	0.8751	0.8756	0.8762	0.8768	0.8774	0.8779	0.8785	0.8791	0.8797	0.8802
3.1	0.4914	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038	7.6	0.8808	0.8814	0.8820	0.8825	0.8831	0.8837	0.8842	0.8848	0.8854	0.8859
3.2	0.5051	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172	7.7	0.8865	0.8871	0.8876	0.8882	0.8887	0.8893	0.8899	0.8904	0.8910	0.8915
3.3	0.5185	0.5198	0.5211	0.5224	0.5237	0.5250	0.5263	0.5276	0.5289	0.5302	7.8	0.8921	0.8927	0.8932	0.8938	0.8943	0.8949	0.8954	0.8960	0.8965	0.8971
3.4	0.5315	0.5328	0.5340	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428	7.9	0.8976	0.8982	0.8987	0.8993	0.8998	0.9004	0.9009	0.9015	0.9020	0.9025
3.5	0.5441	0.5453	0.5465	0.5478	0.5490	0.5502	0.5514	0.5527	0.5539	0.5551	8.0	0.9031	0.9036	0.9042	0.9047	0.9053	0.9058	0.9063	0.9069	0.9074	0.9079
3.6	0.5563	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.5670	8.1	0.9085	0.9090	0.9096	0.9101	0.9106	0.9112	0.9117	0.9122	0.9128	0.9133
3.7	0.5682	0.5694	0.5705	0.5717	0.5729	0.5740	0.5752	0.5763	0.5775	0.5787	8.2	0.9138	0.9143	0.9149	0.9154	0.9159	0.9165	0.9170	0.9175	0.9180	0.9186
3.8	0.5798	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899	8.3	0.9191	0.9196	0.9201	0.9206	0.9212	0.9217	0.9222	0.9227	0.9232	0.9238
3.9	0.5911	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.6010	8.4	0.9243	0.9248	0.9253	0.9258	0.9263	0.9269	0.9274	0.9279	0.9284	0.9289
4.0	0.6021	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117	8.5	0.9294	0.9299	0.9304	0.9309	0.9315	0.9320	0.9325	0.9330	0.9335	0.9340
4.1	0.6128	0.6138	0.6149	0.6160	0.6170	0.6180	0.6191	0.6201	0.6212	0.6222	8.6	0.9345	0.9350	0.9355	0.9360	0.9365	0.9370	0.9375	0.9380	0.9385	0.9390
4.2	0.6232	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325	8.7	0.9395	0.9400	0.9405	0.9410	0.9415	0.9420	0.9425	0.9430	0.9435	0.9440
4.3	0.6335	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425	8.8	0.9445	0.9450	0.9455	0.9460	0.9465	0.9469	0.9474	0.9479	0.9484	0.9489
4.4	0.6435	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522	8.9	0.9494	0.9499	0.9504	0.9509	0.9513	0.9518	0.9523	0.9528	0.9533	0.9538
4.5	0.6532	0.6542	0.6551	0.6561	0.6571	0.6580	0.6590	0.6599	0.6609	0.6618	9.0	0.9542	0.9547	0.9552	0.9557	0.9562	0.9566	0.9571	0.9576	0.9581	0.9586
4.6	0.6628	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712	9.1	0.9590	0.9595	0.9600	0.9605	0.9609	0.9614	0.9619	0.9624	0.9628	0.9633
4.7	0.6721	0.6730	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803	9.2	0.9638	0.9643	0.9647	0.9652	0.9657	0.9661	0.9666	0.9671	0.9675	0.9680
4.8	0.6812	0.6821	0.6830	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893	9.3	0.9685	0.9689	0.9694	0.9699	0.9703	0.9708	0.9713	0.9717	0.9722	0.9727
4.9	0.6902	0.6911	0.6920	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981	9.4	0.9731	0.9736	0.9741	0.9745	0.9750	0.9754	0.9759	0.9763	0.9768	0.9773
5.0	0.6990	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.7050	0.7059	0.7067	9.5	0.9777	0.9782	0.9786	0.9791	0.9795	0.9800	0.9805	0.9809	0.9814	0.9818
5.1	0.7076	0.7084	0.7093	0.7101	0.7110	0.7118	0.7126	0.7135	0.7143	0.7152	9.6	0.9823	0.9827	0.9832	0.9836	0.9841	0.9845	0.9850	0.9854	0.9859	0.9863
5.2	0.7160	0.7168	0.7177	0.7185	0.7193	0.7202	0.7210	0.7218	0.7226	0.7235	9.7	0.9868	0.9872	0.9877	0.9881	0.9886	0.9890	0.9894	0.9898	0.9903	0.9908
5.3	0.7243	0.7251	0.7259	0.7267	0.7275	0.7															

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- ⁱⁱⁱ NCTM, *Historical Topics for the Mathematics Classroom*, 105.
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- ^v Bob Burn, "Napier's Logarithms," *Mathematics in School* 27 (1998): 32-33, accessed June 25, 2013, www.jstor.org/stable/30211872.
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- ^{ix} Cajori, "History of Exponential and Logarithmic Concepts," 13.
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Curriculum Unit Title	Why We Need Logarithms: From Early Navigation to Nuclear Meltdown	Author	Nancy Rudolph
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KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Being able to compute fluently means making smart choices about which tools to use and when to use them. Knowing the reasonableness of an answer comes from using good number sense and estimation strategies. Mathematics is built on reason and always makes sense. Mathematics can be used to solve problems outside of the mathematics classroom.

Numbers or objects that repeat in predictable ways can be described or generalized. An operation can be “undone” by its inverse. The inverse relationship between exponents and logarithms can be used to solve problems involving logarithms and exponents. Rules of arithmetic and algebra can be used together with notions of equivalence to transform equations so solutions can be found.

ESSENTIAL QUESTION(S) for the UNIT

How can change be described mathematically? How are patterns of change related to the behavior of functions? How do mathematical models/representations shape our understanding of mathematics? How can exponential and logarithmic models be expressed using inverse relationships to find solutions?
What makes a strategy both *effective* and *efficient*? Is your plan working? Do you need to reconsider what you are doing? How do you make sense of different strategies? How do you determine their strengths and weaknesses?

CONCEPT A

Historical perspective of logarithms and their properties

CONCEPT B

Inverse Relationships in Functions

CONCEPT C

Solve everyday problems that can be modeled using exponential and logarithmic functions.

ESSENTIAL QUESTIONS A

What are the connections between properties of logarithms and exponent rules? How can an expression be written in different but equivalent forms to reveal and explain different properties of the function?

ESSENTIAL QUESTIONS B

How is the inverse relationship between exponents and logarithms used to solve problems involving exponents and logarithms?

ESSENTIAL QUESTIONS C

How can exponential or logarithmic functions model relationships between two quantities?

VOCABULARY A

Arithmetic progression/sequence
Geometric progression/sequence
Exponents
Logarithms

VOCABULARY A

Inverse

VOCABULARY A

Radiation, radioactive decay, nuclear fission
Half-life
Carbon-dating

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

Base 10 Logarithm Table