A Modeling Unit on Quadratics

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Rationale

Quadratics has been the one major topic in high school mathematics where year after year I clearly underperform as an educator in teaching the majority of my students this specific set of mathematical skills and concepts. My students generally do fine on their unit test but subsequent testing on the topic reveals that the skills they used months earlier were not stored in their long-term memories. I believe that at the heart of this lack of transfer from short-term to long-term memory is the simple truth uncovered by brain research that the only way this transfer takes place is if students can make sense of what they are learning.[1] I believe that infusing modeling into this unit can help students make sense of and see value in the skills and concepts they are learning and hopefully promote transfer into long-term memory. The modeling focus will be centered primarily on two real world systems: launched fireworks and a very deep hole.

Student population

This unit will be taught to two different freshmen sections at John Dickinson High School during the 2012-2013 school year. During the fall semester it will be taught to an Integrated Math 2 Honors section; during the spring semester it will be taught to an Integrated Math 2 College Prep section. Dickinson has a 62% minority student population, and 61% of its students receive free or reduced lunch services.[2] 21% of Dickinson freshmen passed the 9th grade DCAS math assessment in the fall of 2012. All students must have passed either Algebra 1 or Integrated Math 1 as a prerequisite for taking Integrated Math 2—where this unit will be presented. Relevant background knowledge includes a strong foundation in linear functions, how to graph functions by plotting points, multiplying multi-digit integers, and representing unknown quantities as variables.

Unit objectives

- Students will be able to convert any quadratic equation into both vertex form and standard form.
- Students will be able to identify any rational x-intercepts, or roots, of any quadratic function.
- Students will be able to identify the vertex of any quadratic function.

- Students will be able to create a quadratic equation if given relevant information about the function: vertex and width; initial velocity, height, and oppositional forces (such as gravity).
- Students will be able to multiply polynomials.
- Students will be able to factor trinomials into two binomials where all coefficients and constants can be represented by integer values.

Standards

The following CCSSM strands represent a non-exhaustive list of standards directly addressed in this unit.[3]

Primary Content Standards

- A-SSE.1. Algebra, Seeing Structure in Expressions: Interpret the structure of expressions. Interpret expressions that represent a quantity in terms of its context (Modeling Standard).
- A-SSE.3. Algebra, Seeing Structure in Expressions: Write expressions in equivalent forms to solve problems. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression (Modeling Standard).
- A-APR.1. Algebra, Arithmetic with Polynomials and Rational Expressions: Perform arithmetic operations on polynomials. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
- A-APR.3. Algebra, Arithmetic with Polynomials and Rational Expressions: Understand the relationship between zeros and factors of polynomials. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.
- A-CED.2. Algebra, Creating Equations: Create equations that describe numbers or relationships. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
- A-REI.4.a. Algebra, Reasoning with Equations and Inequalities: Solve equations and inequalities in one variable. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
- F-IF.7.a. Functions, Interpreting Functions: Analyze functions using different representations. Graph linear and quadratic functions and show intercepts, maxima, and minima.

- F-IF.8. Functions, Interpreting Functions: Analyze functions using different representations. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
- F-IF.8.a. Functions, Interpreting Functions: Analyze functions using different representations. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
- F-BF.1. Functions, Building Functions: Build a function that models a relationship between two quantities. Write a function that describes a relationship between two quantities (Modeling Standard).
- F-LE.1. Functions; Linear, Quadratic, and Exponential Models (Modeling Cluster): Construct and compare linear, quadratic, and exponential models and solve problems. Distinguish between situations that can be modeled with linear functions and with exponential functions.

Secondary Content Standards

- N-Q.1. Number and Quantities, Quantities: Reason quantitatively and use units to solve problems. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and origin in graphs and data displays.
- N-Q.2. Number and Quantities, Quantities: Reason quantitatively and use units to solve problems. Define appropriate quantities for the purpose of descriptive modeling.
- A-SSE.1.a. Algebra, Seeing Structure in Expressions: Interpret the structure of expressions. Interpret parts of an expression, such as terms, factors, and coefficients.
- A-SSE.2. Algebra, Seeing Structure in Expressions: Interpret the structure of expressions. Use the structure of an expression to identify ways to rewrite it.
- A-SSE.3.a. Algebra, Seeing Structure in Expressions: Write expressions in equivalent forms to solve problems. Factor a quadratic expression to reveal the zeros of the function it defines.
- A-SSE.3.b. Algebra, Seeing Structure in Expressions: Write expressions in equivalent forms to solve problems. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
- A-APR.4. Algebra, Arithmetic with Polynomials and Rational Expressions: Use polynomial identities to solve problems. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
- A-CED.1. Algebra, Creating Equations: Create equations that describe numbers or relationships. Create equations and inequalities in one variable and use them to

solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

- A-REI.2. Algebra, Reasoning with Equations and Inequalities: Understand solving equations as a process of reasoning and explain the reasoning. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
- A-REI.4.b. Algebra, Reasoning with Equations and Inequalities: Solve equations and inequalities in one variable. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation.
- A-REI.7. Algebra, Reasoning with Equations and Inequalities: Solve systems of equations. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ and the circle $x^2 + y^2 = 3$.
- F-IF.5. Functions, Interpreting Functions: Interpret functions that arise in applications in terms of the context. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function* (Modeling Standard).
- F-IF.9. Functions, Interpreting Functions: Analyze functions using different representations. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*
- F-BF.3. Functions, Building Functions: Build new functions from existing functions. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Primary Mathematical Practice Standards

While I believe all eight mathematical practices are present during this unit, these five are definitely at the forefront of what is happening this unit:

- MP-1. Make sense of problems and persevere in solving them.
- MP-2. Reason abstractly and quantitatively.
- MP-4. Model with mathematics.
- MP-6. Attend to precision.
- MP-7. Look for and make use of structure.

The Unit

#	Activity	Time (min.)	Mathematical Skill	Relationship to modeling	Resources
1	Fireworks unit problem presented	30		System introduced	Fireworks video
2	Hole unit problem presented	30		System introduced	Hole video Act1 and Act2
3	*October Sky* movie clip and activity	45	Build a quadratic equation from a linear framework	Discover the effect gravity has on a system with a free-falling object	Movie clip and data table
4	"Will It Hit the Hoop?" activity	40	Develop understanding of vertex form of a quadratic function and recognize its value	The Fireworks unit problem asks in context for a vertex; students will convert from standard to vertex form and then will interpret the *h* and *k* in context.	"Will it Hit the Hoop?" by Dan Meyer with Geobebra
5	Skill building activities		Multipyling polynomials; finding extrema via vertex form	Finding the max height of the rocket will be done via conversion to vertex form	
6	The Quadratic Formula via completing the square	45	Practice completing the square with given standard form quadratic functions, then generalize	Finding the roots of a quadratic function will be particularly useful for finding when the firework lands.	none needed
7	Modified Fireworks unit problem	60	Practice converting from standard to vertex form; practice finding x-intercepts.	A simpler version of the unit problem with more palatable solutions.	IMP2 Textbook

8	Fireworks unit problem solved	75	Practice converting from standard to vertex form; practice finding x-intercepts.	Take this system through the modeling cycle with the ultimate goal of using our model to make contextual decisions.	IMP2 Textbook
9	Conversion and comparison of vertex and standard forms	40	Convert from vertex to standard form and use algebra to compare coefficients of x^2, x and constant term	Develop skills to make analyzing quadratics in standard form easier than converting to vertex form.	none needed
10	Finding the vertex via the Quadratic Formula	40	Students rediscover $h = \frac{-b}{2a}$ and $k = f(h)$.	Develop skills to make analyzing quadratics in standard form easier than converting to vertex form.	none needed
11	Hole unit problem solved	90	Practice creating quadratic and linear equations and solving a non-traditional system of equations	Take this system through the modeling cycle with the ultimate goal of using our model to make contextual decisions.	Hole video Acts 1 and 2, and solution page

Activities

1 Fireworks unit problem presented

Our first modeling system involves finding out information about fireworks. I will start off by showing them a simple fireworks video and then asking some basic questions such as, "When do the fireworks explode?", "Why do the pyrotechnics wait for the fireworks to be at their peak before exploding?", "Do you think we could create a formula that models the height of a firework after t seconds?"[4] The students are given the facts about a specific fireworks launch with an initial velocity (V_i) of 100 feet per second (the assumption, not stated, is that this initial velocity is its vertical initial velocity) from a platform elevated 224 feet above ground level (this system was inspired by, but modified from our textbook: *Interactive Mathematics Program: Year 2* (IMP2)).[5] The students are asked to determine three things: the ideal time for the fuse to be lit before detonating to be seen by the most people (i.e. when does the fireworks reach its maximum height after launch), the height of the fireworks be at this ideal time, the number of seconds after their launch will the fireworks hit the ground.

2 Hole unit problem presented

The second modeling system involves a video of two construction workers dropping a large rock down a hole which we hear hit the ground several seconds after its release.[6] Students pose a series of questions to the video, but as a class we choose to focus on the question, "How deep is the hole?"

Both problems are introduced at the beginning of the unit. And in both cases students answer each question with an answer they know will be too large, an answer they know will be too small, and finally with their best guess to the true answer—these answers are recorded on a SMART Notebook file to be shown to the class again after the problems are solved.

After being introduced to the hole problem students try to solve the three problems posed. They also are tasked with creating a list of things they currently do not know but will need to know to solve the problems. It only takes a few minutes of brief engagement before students begin to see their own mathematical shortcomings. It should be noted that these students all have a strong background with linear functions and have modeled linear systems several times. All the students come in with the knowledge that within a linear system, $y = a + bx$, where a is the starting point (the value of your dependent variable when your independent variable is zero) and b is the rate of change (change in y in terms of x).

Students will now begin to engage with the second unit problem.

- Students guess an answer too high, too low, and their actual guess to the hole's depth.
- Students will brainstorm in their groups about information they think they will need to solve this problem.
- The teacher will have prepared information concerning the speed of sound, the time from the rock's release until we heard the crash (this will actually be shown to a class as a separate video, Figure 1), and the location of the construction site. The teacher will ask each group which information they think they need and will provide it to that group provided he has it. Groups will be able to ask the teacher later for additional information if they think they missed something. (This information gathering step may also be done as a whole-class activity.)

Figure 1: Screenshot from Act 2 of the Hole unit problem[7]

That is all we will do with this activity originally.

These two unit problems are referred to throughout the rest of the unit and are presented as goal-oriented challenges. Students must engage with sustained effort to eventually conquer these challenges. This structure is deliberately developed in an attempt to access students' eudaimonic pleasure and stimulate their motivation to learn.[8]

3 October Sky movie clip and activity

After breaking from the original unit problem students will be shown a clip from the movie *October Sky* showing a brief two-minute scene of the protagonist successfully launching a rocket high into the air.[9] The students will be given the initial velocity of 642 ft/sec (and will assume the initial height of 0 on their own). They will be asked to model the rocket's height for its first 10 seconds of flight using a linear model. It is expected that students will voluntarily share their linear models of:

$$h(t) = 642t.$$

Students will be asked to reflect on why their linear models are insufficient. After two minutes of think-time the teacher will prompt the question, "How high will the rocket be after two minutes?" and, "Does this height validate your intuition about where the rocket will be after two minutes?" Students typically recognize that eventually the rocket has to come down and that is currently not modeled by their linear function. They hopefully conclude their linear function is insufficient, that they have linearized the non-linear, and that a different function to model the rocket's behavior should be sought.[10] A table is revealed with the actual heights of the rocket after t seconds (See Appendix A for finished table. The data is fabricated to fit perfectly into the quadratic function I want them to come up with.) Students then examine the difference in their linearly predicted data to the actual data. Students examine the difference at each time interval (each second) and begin to search for patterns in the data. If a student recognizes each difference has a common factor of -16, he/she will eventually be asked to reveal their finding. If no student recognizes the greatest common factor after several minutes then the teacher will ask the prompting question "Is there a common factor in your difference data?" After students factor out a -16 from their difference data they will recognize that their data was off from the actual data by a difference of $-16t^2$. Students are now able to construct an equation that models the actual height of the rocket by adding $-16t^2$ to their original, linear model:

$$h(t) = 642t + -16t^2.$$

Students will check their new function by substituting in random times to see if their heights' match the data from the table. Students are then prompted by the teacher with two new questions: "Where does that $-16t^2$ come from?" and "When would our original model have been accurate?" For the closure activity students attempt to create an equation to model our first unit problem, the firework one, with the inclusion of the effect of gravity.

4 "Will It Hit the Hoop?" activity

The next activity will focus on a prearranged model prepared by the teacher through the software program Geogebra. Computers will be needed for individual students (or at least one computer per group if a one-to-one ratio cannot be procured). Several Geogebra files will be preloaded on each computer. Teachers will instruct students to open the first file which is a movie file showing a man shooting a basketball towards a basket.[11] The video cuts off right after the ball begins descent. Students are prompted to write down if they think the ball goes in. Often one or more students will now, or later in the activity, bring up the fact we cannot know if the ball will go in because we are looking at a three-dimensional activity (making a basket) through a two-dimensional lens (one perspective through a camera). This question presents a great opportunity to talk about parallax dilemma as a class. After making predictions students will open a Geogebra file (see image below), that has the vertex form of a parabola preloaded:

$$y = a(x - h)^2+k,$$

with sliders set for *a*, *h*, and *k* (originally all set to zero). Students must then adjust the values on these three sliders to have the parabola originally trace over, but then model the flight of the ball.

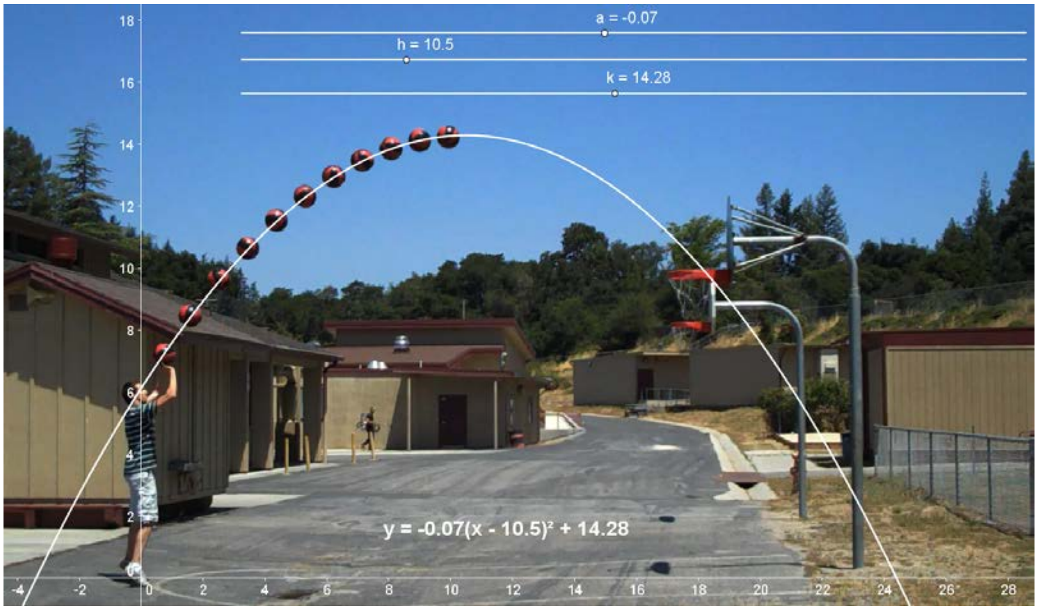


Figure 2: Screenshot from Dan Meyer's "Will It Hit the Hoop?" activity[12]

Students then use the power of their graphical model to re-predict whether or not the ball goes into the basket. After students have re-predicted the teacher shows the full video of the shot showing that the ball does in fact go into the basket. Students repeat this entire process another four times for four new videos of basketball shots each accompanied with a matching Geogebra file. The teacher has everyone share each of the equations they got for their model and records them on the board for the class to see. There is a discussion about the effects that each slider has on the parabola. Students will come to the conclusion (scaffolding may be necessary) that their *k* value represents the highest point of the parabola or the highest point of the basketball's path. Students will also conclude that their *h* value represents how far over their maximum height on the parabola was (so if *k* reached a maximum of 14 at an x-value of 7.5, it would mean *h* was 7.5 on their slider—so there is a direct correlation between the x-value of their vertex and the value of *h*). Students will also discuss the effects that changing the value of *a* had on their parabola, specifically that the further its value was from one the "skinnier" or steeper the parabola became, and also that positive values of a gave us a "smiley-face" or concave-up parabola whereas negative values of a gave us a "frowney-face" or concave-down parabola. The purpose of this activity is multi-faceted. Primarily I want my students to see the power of the structure of the vertex-form of a parabola and how simple it is to transfer from graph to equation in this form, and how much information we can immediately obtain from a quadratic written in this form. Secondly I want my students to become familiar with the effects we see in this form so they can apply it to non-quadratic functions (i.e. what does adding a constant do to a linear function? a trigonometric function?) The goal is that they will some power in the structure of mathematics through these similarities. And, of course, I want my students to buy into the fact that math models the real world. I think most math teachers have bought into that fact and assume students have as well because we repeatedly tell them math does in fact model the real world, but I think students need to experience this wonderful truth for themselves to truly embrace and take ownership of it which I believe will lead to much improved student efficacy and mathematical agency.

After the basketball modeling activity students will engage in two more activities involving geometric transformations where they will have to sketch graphs given a quadratic equation represented in vertex-form, and conversely they will have to create equations in vertex-form based on a given set of graphs of parabolas. These activities will be presented devoid of context to see if students were able to abstract out from the basketball context. Students will then move to situations where they are given the vertex of a parabola along with another point on that parabola and must find the value of *a*, and then find the equation of the parabola in vertex form. One context later in this unit where this math will be applied is with predicting whether or not a hit is a homerun given the vertex point of the ball and the height and distance of the homerun fence (students must intuitively come up with a second point on the parabola such as (0, 0) or (0, 3) to represent the location of the ball when the batter hit it). Students also complete an activity where they are given a parabola and its equation and examine the symmetrical nature of the parabola by substituting equidistant x-values into the equation and getting the same y-values; this symmetry comes into play later when they find h (or the x-value of the vertex) given a parabola's x-intercepts.

5 Skill building activities

Students will then engage in several activities that will teach them the following skills:

- How to find the maximum area of a rectangle using a quadratic's vertex form.
- How to multiply polynomials.
- How to maximize the volume of a rectangular prism using a quadratic's vertex form.
- How to find the height of a non-right triangle by using the triangle's altitude to break the base into two parts and setting both parts equal to the altitude.
- How much a company should charge for a product balancing profit and product demand.

6 The Quadratic Formula via completing the square

Students finally move to the skill of converting from standard form to vertex form, drawing on activities from earlier this unit of multiplying polynomials through drawing a rectangle, students are introduced to the idea of factoring trinomials in a skill commonly referred to as "completing the square" (even the formal Common Core State Standards for Mathematics (CCSSM) refer to this skill by name multiple times). After converting the quadratic equation to vertex form students are asked to find the x-intercepts of the function. Students are asked to contemplate what x-intercepts are and are prompted to decide how they would conceptually then find the x-intercepts of a quadratic function in vertex form. Students determine they must substitute zero for their y-value then solving for x is intuitive (with the reminder that each number has a positive and negative square root). Students need to be able to find the x-intercepts of a quadratic function to solve parts of both unit problems. Students eventually use their skills of completing the square and finding the x-intercepts of a quadratic function to find the x-intercepts of any quadratic in standard form—which of course yields the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(The formula is accompanied by a song to serve as a pneumonic for remembering it.) It is a formula they may choose to use, but certainly do not have to, to help solve both unit problems.

7 Modified Fireworks unit problem

Students then look at some new modeling problems including a problem in the same vein as their fireworks unit problem with a different starting height and initial velocity (and with more palatable solutions).[13]

8 Fireworks unit problem solved

Students now go on to solve the first unit problem introduced approximately twelve classes prior.

- System: A firework is launched off a platform 160 feet in the air with an initial vertical velocity of 92 feet per second somewhere on Earth. The firework has enough mass for terminal velocity to be negligible.
- Problems/questions: How long does it take for this firework to reach its maximum height? How high is its maximum height? How long does it take for this firework to reach the ground?
- Formulate original mathematical model: Originally many students put, h = 92t + 160, where h is the firework's height in feet and t is the time after launch in seconds. Some students put h = (92 – 32)t + 160, with 32 attached to their linear rate and coming from the effect of gravity on the firework.
- Simplification/analysis: Many students realized that their original models would never have a height of zero and would therefore never land and were therefore inaccurate mathematical models.
- Formulate mathematical model, round 2: After the activity featuring the clip from *October Sky* many students were able to change their initial linear model to a more appropriate: $h = 160 + 92t – 16t^2$.

- Simplification/analysis, round 2: Students found confidence in their model, but were unable to answer the questions stated above other than giving the range that the maximum height is slightly less than three seconds and the firework hits the ground slightly after seven seconds.
- Prediction: Students can predict the height of the firework after any number of seconds. Through guess-and-check they can find when the rocket reaches its vertex and how high its vertex is; they can also predict very closely when that rocket hits the ground.
- Formulate mathematical model, round 3: Students rewrite their mathematical model in vertex form.
- Simplification/analysis, round 3: Students apply knowledge of vertex form to correctly identify the vertex of the parabola and thereby identifying how long it takes the firework to reach its maximum height and what the firework's maximum height will be. From vertex form students find both x-intercepts of the parabola and identify one as extraneous, therefore identifying how long it takes the firework to hit the ground.
- Prediction, final (after next activity): After a discussion on the structures comparing vertex form and standard form of quadratics students create a generalized equation that models how long it will take a firework to reach its maximum height, what that maximum height will be, and how long it will take the firework to land with an initial height of H_i feet and an initial velocity of V_i feet per second (and launched on Earth).

9 Conversion and comparison of vertex and standard forms

After solving the first unit problem the teacher will prompt the class to rewrite $y = a(x - h)^2 + k$ in standard form. Students will then be given time to compare standard form and vertex form rewriting h and k from vertex form in terms of a, b, and c from standard form. A class discussion on each coefficient will take place on the three equations discovered via the comparison of functions:

$$a_{\text{vertex form}} = a_{\text{standard form}};\ h = \frac{-b}{2a};\ k = c - ah^2.$$

The class discusses what each of these mean and why they are useful (primarily for figuring out lots of information about a parabola without having to engage in the tedious process of converting to vertex form). Students then re-examine the unit problem using the shortcuts we discovered in this activity. Students find confidence in their near conversion-equations as they see they got the same answers to the unit problem as they got by completing the square and converting to vertex form. Students are then asked to solve the same three questions they were asked in the unit problem but are now given an initial height of H_i feet and initial velocity of V_i feet per second.

10 Finding the vertex via the Quadratic Formula

The next day the students do a follow-up activity where they re-discover that $h = \frac{-b}{2a}$ this time by using the Quadratic Formula coupled with the fact that a parabola is symmetrical about the axis x = h (discovered in a prior activity described above). Students discover that h is the midpoint of the two x-intercepts which is easy for them to apply once they have numerical values for the intercepts but often must be prodded to find the generalization:

$$h = \frac{\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}}{2},$$

which simplifies nicely to be of course, $h = \frac{-b}{2a}$, as the discriminants from each intercept cancel out when added together.

11 Hole unit problem solved

Students now attempt to conquer the second unit problem, the depth of the hole.

- System: A video of a hole in Austria and someone dropping a big rock into it followed by the sound of that rock hitting the bottom of the hole.
- Problems/questions: How deep is the hole?
- Formulate original mathematical model: Students originally model the height of the hole (where the video shows us 11.97 seconds between the rock entering the hole (with no initial velocity placed upon it) and us hearing the crash), as $d = 16t^2$, where d is the depth of the hole and t is the number of seconds the rock falls; students recognize both the initial height and velocity are zero leaving them no first degree term or constant. After examining the video more thoroughly where the video shows us 11.97 seconds between the rock entering the hole (with no initial velocity placed upon it) and us hearing the crash, students change their model to $d = 16(11.97)^2$ (some students use negative 16 here; there is often a discussion at some point as to whether or not it should be positive or negative, the students usually conclude that it should be positive as distance cannot be negative).[14]
- Prediction: Students use their original model of $d = 16t^2$ to predict how deep any hole as if were given the time it took for the rock to hit the ground, which is actually correct (not accounting for factors such as terminal velocity). Students use this model to incorrectly predict that this hole is 2292.5 feet deep. (Students incorrectly assumed that they were given t when in fact they were given a different data point.)
- Simplification/analysis: Students think their answer of 2292.5 feet is correct as it seems reasonable to all of them. The teacher through a series of questions and perhaps a re-watching of the video lead students to analyze what actually happens

at 11.97 seconds; students then realize that the rock was falling for fewer than 11.97 seconds and that we must consider the time it took for the rock to hit the ground and the time we heard it; they recognize 11.97 represents the time from drop to the time we heard the rock impact with the ground. Students also recognize they need the speed of sound which I provide for them as 1116 feet per second.

- Formulate mathematical model, round 2: Students begin identifying more variables and coming up with more equations: F = Rock's fall time; S = seconds from crash until we hear it; T = time from drop until we hear crash = 11.97; F + S = T; F + S = 11.97; F = 11.97 – S; distance rock falls = $16F^2$; distance sound travels = 1116S.
- Simplification/analysis, round 2: Students examine the equations and variables they have and try to make sense of them all and try to relate them all to one another. Taking this information to the next step is surprisingly difficult for students. The teacher will often offer the prompt if not already done so, to draw a sequential picture identifying exactly what is happening. Other prompts might need to be given before students recognize that the distance the rock falls is the same as the distance sound travels.
- Formulate mathematical model, round 3: $16F^2$=1116S, therefore $16(11.97 – S)^2$= 1116S (students recognize to solve an equation it needs to be eventually written in one variable).
- Simplification/analysis, round 3: Students now simplify and set one side equal to zero and almost every student then applies the Quadratic Formula to solve for *S* (a few still complete the square). Students substitute their value of *S* into d = 1116S to find the distance the sound travelled and therefore figure out the depth of the hole. Students are shown the correct answer via the hole maker's own conclusion (see Appendix C). (Students will have to convert from feet to meters after they are shown the answer.)
- Prediction, final: Students are able to predict the depth of the hole easily if given *S* or *F* and students verbalize that fact here. Students though are asked to generalize their formula if *T* is unknown—to solve for *d* in terms of *T*.

Unit conclusion post-modeling

After solving this second unit problem students spend a few activities learning how to factor from non-perfect squares (completing the rectangle); algebra tiles can be used as a manipulative here originally. Students are then asked to find the x-intercepts of a quadratic equation written in factored form. Then students must contemplate about when to attempt to use each of the various forms they have learned about in this unit to examine quadratic functions: vertex form, standard form, and factored form. Students final mathematical task of the unit is to complete a chart finding standard form, factored form, vertex form, the parabola's vertex, and x-intercepts when given limited information about the quadratic function (see Appendix D).[15]

Appendix A

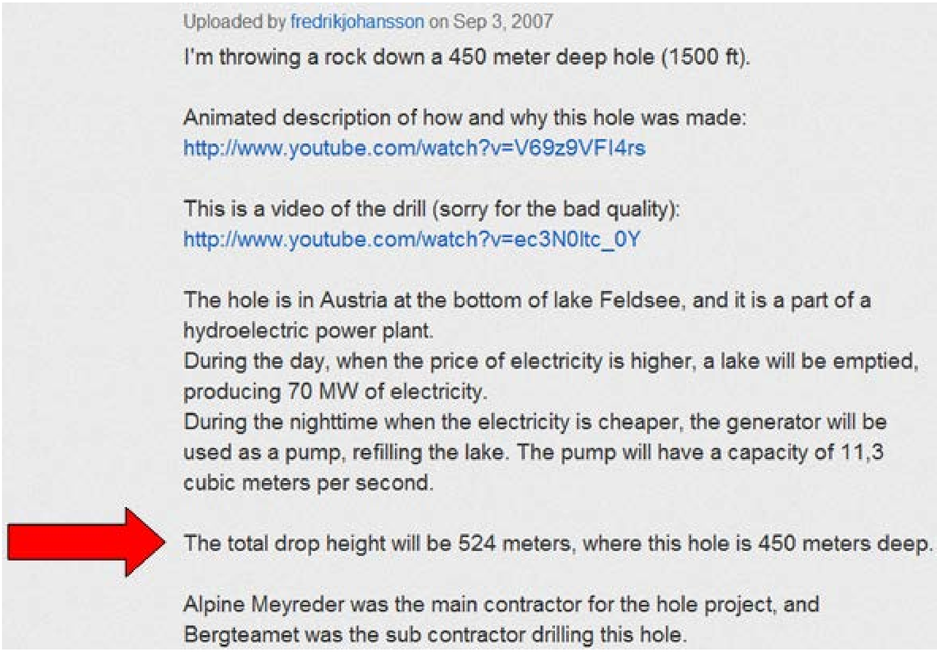
The fully completed table from their *October Sky* data and analysis looks like this (with the only information provided by the teacher being the actual height of the rocket data):

	Linear (Prediction)	Actual height of rocket		
Time	Feet	Feet	Difference	Difference Divided by -16
0	0	0	0	0
1	672	656	-16	1
2	1344	1280	-64	4
3	2016	1872	-144	9
4	2688	2432	-256	16
5	3360	2960	-400	25
6	4032	3456	-576	36
7	4704	3920	-784	49
8	5376	4352	-1024	64
9	6048	4752	-1296	81
10	6720	5120	-1600	100

Appendix B

The vocabulary words they are given are: quadratic functions, standard form, vertex form, factored form, parabola, vertex, intercepts, quadratic equations.

Appendix C


Uploaded by fredrikjohansson on Sep 3, 2007

I'm throwing a rock down a 450 meter deep hole (1500 ft).

Animated description of how and why this hole was made:
http://www.youtube.com/watch?v=V69z9VFI4rs

This is a video of the drill (sorry for the bad quality):
http://www.youtube.com/watch?v=ec3N0ltc_0Y

The hole is in Austria at the bottom of lake Feldsee, and it is a part of a hydroelectric power plant.
During the day, when the price of electricity is higher, a lake will be emptied, producing 70 MW of electricity.
During the nighttime when the electricity is cheaper, the generator will be used as a pump, refilling the lake. The pump will have a capacity of 11,3 cubic meters per second.

The total drop height will be 524 meters, where this hole is 450 meters deep.

Alpine Meyreder was the main contractor for the hole project, and Bergteamet was the sub contractor drilling this hole.

(I added the arrow[16].)

Appendix D

Standard form	Factored form	x-intercepts	Parabola's vertex	Vertex form
$y = x^2 + 4x - 5$				
$y = x^2 - 7x - 18$				
				$y=(x-1)^2 - 25$
	$y = (x - 3)(x + 7)$			
	$y = (x + 2)(x - 6)$			
		(-4, 0) and (6, 0)		
$y = x^2 + 6x - 16$				
		(5, 0) and ( )	(1, -16)	
$y = x^2 - 25$				
	$y = (x - 3)^2$			

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Curriculum Unit Title: Modeling with Quadratics

Author: Michael Reitemeyer

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Students will know how to convert a quadratic from standard form to vertex form. Students will know how to find the x-intercepts and vertex from a quadratic function in standard form. Students will know how to find the intersection point(s) of linear and quadratic functions.

ESSENTIAL QUESTION(S) for the UNIT

How can we model contexts with a constant acceleration?

CONCEPT A	CONCEPT B	CONCEPT C
Quadratic functions stem from linear functions.	How to convert a quadratic from standard form to vertex form.	The vertex of a standard form quadratic is (-b/(2a), f(-b/(2a)))
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
How does the structure of a quadratic function compare to the structure of a linear function?	Why do we want a quadratic in vertex form? How do we convert a quadratic into vertex form?	Can we determine the vertex of a quadratic without converting it into vertex form?
VOCABULARY A	**VOCABULARY A**	**VOCABULARY A**
linear, quadratic	complete the square, polynomial, factor	vertex, standard form, vertex form, x-intercepts

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

Interactive Mathematics Program Year 2, *October Sky*

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[12] Ibid.

[13] Fendel, "Another Rocket," 312.

[14] "HoleAct2," http://www.youtube.com/watch?v=ml3KGb1nQqA&feature=youtu.be.

[15] Fendel, "Standard Form, Factored Form, Vertex Form," 351.

[16] "Dropping a rock down a 1500 foot deep hole," September 3, 2007, video clip, accessed December 1, 2012, YouTube, http://www.youtube.com/watch?v=ml3KGb1nQqA&feature=youtu.be.