Conceptual Understanding of Fractions: The "Why" Behind the "How"

Chuck Sanders

Introduction/Rationale

In the 2013-14 school year Delaware's largest school district, Christina, enrolled 16,721 students which amounts to 80.7% of the students who reside in the district. The other 19.3%, or 3,997 students who reside in the district attended non-public schools. The district is comprised of a section of Wilmington, Delaware's largest city, and the city of Newark, Delaware some 14 miles to the south. There are three high schools, four middle schools and 18 elementary schools. The elementary schools are configured K-5. I am a teacher at McVey Elementary School in Newark. It is situated among a quiet community of small detached homes just off the I-95 corridor and within a mile of the University of Delaware main campus. The 2013-14 school year was comprised of 445 students that included three classrooms of 72 fifth graders. I was one of four educators on the fifth grade team. http://profiles.doe.k12.de.us

Every year, I am very proud of the Herculean efforts of the 5th grade team and the collaboration that helps ensure student growth in state DCAS scores and accomplishes/maintains the building making Annual Yearly Progress (AYP). For instance, in mathematics the number of proficient fifth grade students grew from 34% to 70% over the course of the year. The data is encouraging, but upon closer examination in our Professional Learning Communities (PLC) during the school year, the vast majority of these same students cannot pass district summative assessments aligned to the Common Core State Standards Math (CCSSM). The only exception to this is our gifted population. As members of a community of teachers, we have observed one another teaching, identified student gaps in learning, and analyzed district assessment data and the DCAS to drive the teaching that goes on in fifth grade classrooms.

There could be several reasons for this. For example, the Delaware Department of Education (DDOE) did not fully implement the Common Core Standards until the academic year 2012-13, so its full effect with regard to all applicable domains has yet to be realized at the fifth grade level. There will not be a fifth grade that has fully benefited from the K-4 CCSSM until the school year 2017-18. A goal of the CCSSM is to level the playing field to produce college and career ready students across the entire nation. These standards can be likened to a staircase. Each step on the stairs is a skill that needs to be learned before stepping up to the next one. Each standard creates a landing on the staircase, or stop along the way toward college and career readiness. CCSSM standards are not the learning themselves. They are opportunities along the way for educators, students and their families to work together. [1]

That being said, my work as teacher has also led me to believe that there is a gap in students' conceptual understanding and the question is why. A conclusion of mine is that many of us ourselves have been taught by rote. Rote, also known as procedural fluency, can be defined as efficient, step by step rules used to solve problems. [2] Teachers themselves are lacking the necessary conceptual understanding for transfer in the execution of their own lessons. When educators have been taught by rote, or do not enjoy teaching mathematics because they themselves do not understand it, the result may be teaching students to memorize, resulting in little to no learning. This is quite likely why students do not retain so much of what they are supposed to know from grade level to grade level according to current standards. Their "understanding" lasts only as long as their short term memory. When taught by rote, student learning results in a misinterpretation of or complete lack of the conceptualization of mathematics. What ensues is the proverbial snowball effect throughout students' school careers. Misunderstanding begets misunderstanding.

I am particularly concerned with my 5th grade students' lack in understanding of the CCSSM Numbers & Operations – Fractions domain. It is at this juncture that my teaching must slow down considerably. Review lessons always morph into acquisition lessons. Again, except for most of our gifted students, there are enormous challenges concerning the conceptual understanding of fractions that I have seen firsthand. So, as any thoughtful educator should, I reflect on my own practice and the observation of and conversations with my colleagues. In doing so, it is my conviction that the teaching of the CCSSM (or any previous set of standards) wrongly begins with or relies too much upon procedural fluency rather than conceptual understanding with regard to the Numbers & Operations-Fractions domain. In other words, students are familiar with the "How" but not the "Why" behind their understanding of fractions. Teachers who themselves have learned mathematics mostly as "rules without reasons" must now learn how to teach for conceptual understanding.[3]

Our students do not remain within the realm of whole numbers forever. For children to understand fractions, fractions cannot be thrust suddenly upon them. Learning is a gradual process with each step firmly rooted in prior experiences. This is why conceptual knowledge is essential as students advance from grade to grade. Learning new concepts depends on what you already know, and as students advance, new concepts will increasingly depend on old conceptual knowledge.[4] If we can convince children that fractions are nothing more than a natural extension of whole numbers, then our chances of success in teaching fractions should increase. Children seem to understand the idea of separating a quantity into two or more parts to be shared fairly among friends. They eventually make connections between the idea of fair shares and fractional parts. [5]

And finally, now that grades four and five are departmentalizing in my building for the 2014-15 school year, I have been designated to teach mathematics to the same. I can now concentrate virtually all my efforts across two grade levels to address the challenge. Using the CCSSM and an emphasis on conceptual understanding, I will design and implement activities with fractions. It is with anticipation I look forward to the fruit of my labors when these same students return the following year as my fifth graders. The press for conceptual understanding should produce students who demonstrate understanding and who can explain the why behind their thinking of fractions. The potential for change can be substantial, even in the short term.

Abstract Mathematics

The series of DTI seminars have shed much light on the concept of number sets and their properties. Number sets and their properties allow us to have all the numbers we would ever care to have and/or need. Each time you move further out from one number set to the next, there is greater capability in mathematics using numbers. Although never explicitly stated, attention should be paid to the progression of these sets of numbers within the CCSSM. Laying a sure foundation of number sets and their sequencing from one set to another is essential for a thorough understanding of fractions (and beyond), which begins in the third grade. When students enter the school arena anywhere from the pre-school years to kindergarten, their formal lessons in mathematics begin with exposure to the natural numbers: the numbers shown via objects and used for cardinality, ordering, etc. Natural Numbers (N), or the counting numbers as they are also called, include numbers from one and upward. Natural numbers are not fractions, nor are they decimals. Within this number set, the commutative, associative, distributive, and identity properties of multiplication can be performed. The same is true for addition except for the identity property. The identity property of addition is not possible because of the absence of zero in this number set. Without zero, it is impossible for any number to keep its sameness using the operation of addition.

Students begin their work with whole numbers (W) as they are introduced to the four basic mathematical operations. Whole Numbers include all of N, the properties found therein, plus zero. Like natural numbers, whole numbers are not fractions, or decimals. The inclusion of zero in this number set allows for the identity property of addition (Ex: $7 + 0 = 7$) and the zero property of multiplication (Ex: $8 \times 0 = 0$). The subtraction of whole numbers though may not always be possible. So, students are required to expand their number sense to that of an understanding of integers. Integers (Z) include the subsets of N and W plus every negative and positive number (except for Q). Like N and W, set Z contains no fractions or decimals. For example, the opposite of 2 is negative 2. Integers, whole numbers and their negatives, allow us to express symbolically what we do not have enough of. This can lead to confusion for some elementary children. I make it a point to ask my students if the commutative property of addition works for subtraction. Their strategies for solving are interesting from the concrete to the abstract. For instance, when subtracting a larger number from a smaller number some students say it cannot be done, others wrongly subtract the subtrahend from the minuend to make it work, while others correctly state you get a negative number. A number line is an excellent tool for representing negative numbers. Use real world applications to solidify student understanding. For instance, a thermometer (a vertical number line), used by many classroom teachers for math and science, can clearly show positive and negative integers throughout the school year. The teaching of financial literacy is another excellent opportunity to teach integers as financial institutions use negative and positive integers to represent debits (negative) and credits (positive).

Rational numbers (Q) include all of N, W and Z. A rational number is a fraction, the ratio of two integers (Ex: 4/5). A rational number is also the quotient of two integers where the divisor is a non-zero number (Ex: 3/4 = .75). Rational numbers can be understood as quantities that sit between integers although being a subset of Q, integers themselves are rational numbers when divided by one (Ex: 12 = 12/1). Another way to look at it is that any quotient or ratio belonging to the set of whole numbers (W) or integers (Z) are rational numbers.

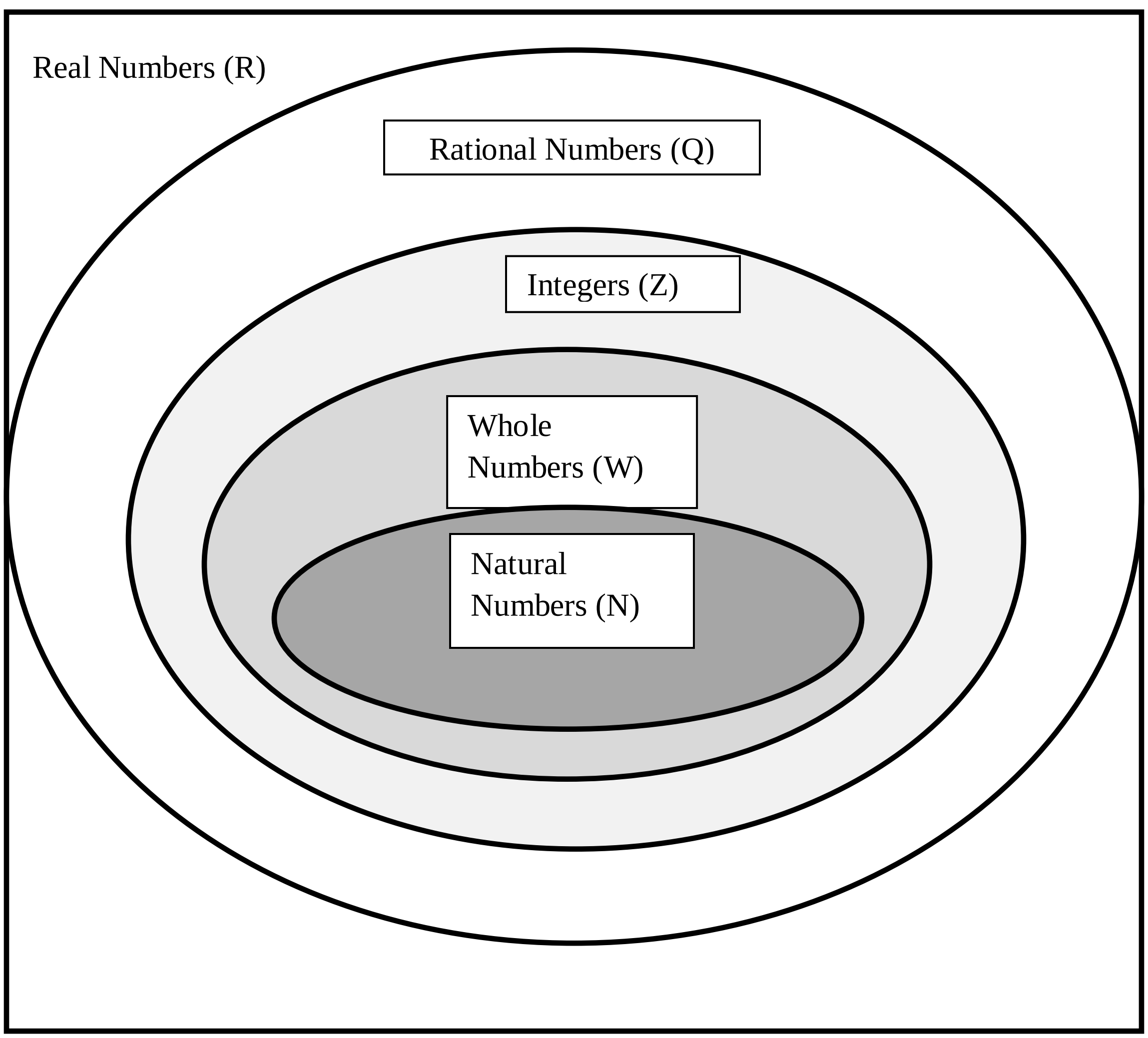


Figure 1

Properties of Operations with Numbers

Along with the above mentioned number sets comes the operation on them. There are four (and no more) at our disposal. These four are sufficient to define all kinds of numbers. The commutative, associate, distributive, and identity properties of addition and multiplication are easy enough to teach, but the challenge is to demonstrate their real importance. It has been challenging for me to discover ways for students to “look long” towards future algebra classes in an attempt to get them to grasp that this is not some trivial, even mundane, disconnected lesson for them to do. Many an attitude and facial expressions suggest, “I got it and can do it, so what’s the big deal?” This left me asking the same question. These properties may indeed seem pointless to an elementary school student.

I share with my students that it is important to be able to identify these properties, but that they will actually use them in their future algebra class. I am thinking that too much emphasis is placed on the operations rather than the properties when teaching. For instance, it would be more appropriate to create skill sets asking students to find equalities but also equalities that show the commutative property. This would work to ensure students understand the value of these properties. From integers, students formally begin work with the pieces of a whole or group and the need arises to be able to represent such. To do so, students must go outside yet another set of numbers and into another identified as rational numbers. It is within the realm of this number set that students are able to work with fractions. Although students at this stage are working with rational numbers, many do bring into the fractions realm a whole number mentality. Children have a tremendously strong mindset about numbers that causes them difficulties with the relative size of fractions. In their experience, larger numbers mean “more”.

As with whole numbers, the operations of addition and subtraction of fractions with common denominators is taught from the beginning. This is accomplished by students by decomposing fractions into sums of fractions using fraction models. Clearly, calculations such as “two sevenths plus three sevenths,” which are not really different from “two apples plus three apples,” should be taught right at the beginning of fractions. But when the addition and subtraction of fractions involve finding a common denominator, they are certainly harder than multiplication and division of fractions, and hence should be taught later. Also, the expansion and reduction of fractions, which is necessary for understanding the notion of the common denominator and finding common denominators, are closely related to the multiplication and division of fractions. [6] Students begin to generate equivalent fractions using models. These visuals allow for them to test and explain their comparisons. This comparison of fractions is extended as students move to fractions that have different numerators and denominators using the <, > and = symbols and offer proof in the form of fraction models.

There are number properties that apply to all real numbers (R). The commutative property allows the changing of the order of addends in addition and likewise the factors in multiplication without changing the sum or product. Students should think “order” for this property. Next is the associative property. Both addition and multiplication can only be done two numbers at a time. So, which two do we associate first? The associate property of addition allows us to group numbers in a sum any way and still get the same answer. Similarly, in multiplication, we can group numbers in a product any way we like without changing the answer. Students should think “grouping” for this property. Next is the distributive property. This property comes into play when both addition and multiplication are involved. Multiplication distributes over addition. Ex: $3(4 + 5)$. With such an expression, the order of operations would have us do the work in the parentheses first and then multiply. However, take the following expression $a(b +c)$. In algebra it may not be possible to add $b + c$, so the distributive property allows you to distribute the a to the b and the a to the c to get $a(b + c) = ab +ac$. Lastly is the identity property. For addition, the sum of zero and any number is that number (Ex: $5 + 0 = 5$). For multiplication, the product of one and any number is that number (Ex: $5 \times 1 = 5$). Students should think “same” for this property. And last is the zero property of multiplication. The product of zero and any number is zero (Ex: $5 \times 0 = 0$). Students should think “zero is the product” for this property.

Background

Just what are fractions? The answer is worth careful consideration because fractions have been defined in many different ways even in my own classroom: a part of a whole number, the part of a set or group, the quotient of the integer c divided by d, and the ratio of c to d. By definition, fractions are rational numbers, or numbers that can be written as fractions with a non-zero denominator. Precision in mathematics is essential and the CCSSM clarifies how fractions should be formally introduced to our third graders: parts of a whole, as numbers on a number line and fractions that are equivalent. With this understanding, extending the understanding of fractions continues in the fourth grade classroom.

Classroom Activities on the Conceptual Understanding of Fractions

Teaching Strategies

Cooperative Learning – with peer assistance as pairs or in small heterogeneous learning communities, students will use a Frayer model, visual, graphic and concrete models to compare and order fractions

Intervention Groups – students will practice learned skills for fluency including differentiated versions

Objectives

Students will construct models to determine equivalence between base ten fractions with denominators of 10 and 100

Students will write to explain/justify their conjectures and share orally

Students will make “good estimates” of area fractions, place correctly on a number line then do the same using number fractions

Students will write inequalities for their fraction cards using the symbols <, >, and =

Students will compare circle (or wheel) fractions for equivalence

Classroom Activities

Activity 1

Comparing Tenths and Hundredths
Duration: 45 minutes

Frayer Model (Appendix C)	Essential Question(s)	Standard
compare, equivalent conjecture, justify	How can I use models to show a fraction with a denominator of 10 is equivalent to a fraction with a denominator of 100? How do I explain in writing my understanding of comparing the numerator and denominator of fractions? Why should mathematical conjectures and their justification be recorded in writing?	CCSS.MATH.CONTENT.4.NF.C.5

Pre-Assessment: Which is Greater?
Present one pair of fractions to students from Appendix B. One fraction should have a denominator of ten and the other a denominator of one hundred. Students must choose which fraction is greater.

Task:
Present another pair of fraction cards from Appendix B, one with a denominator of 10 and the other with a denominator of 100. The students' task is to decide which fraction is greater. Students are to explain in writing why they think this is so and then test their choice using any model(s) they wish to use. Students should write a description of how they made their test and whether or not it agreed with their choice. If incorrect, have students try and state in their explanations what was wrong with their thinking.

Post Assessment:
Present the same pair of fractions from the pre-assessment. Student conclusions should also include an understanding that one way fractions can be compared is when they have a common denominator.

Fluency Activity for Lesson 1: A Sum of One
Cut out and shuffle the decks of tenths and hundredths separately. Place the tenths cards face down in a row on top. Do likewise with the hundredths cards as the bottom row. Students take turns turning over a card from each row in order to find a sum of one whole. If students do not have a match, or if they are unsure, the cards are turned upside down to be used again. The game is over when all the cards are used. The player with the most cards wins.

Differentiated Activity for Lesson 1
Have students discover what other combinations of fractions equal one whole using three or more cards.

Adapted from Van de Walle
Activity 16.10, page 305

Activity Two

About How Much?
Duration: Two 45-minute periods

Frayer (Appendix C)	Essential Question(s)	Standard

estimate, classify, compare, equivalent, number line	How can I use a number line to place and compare fractions?	CCSS.MATH.CONTENT.4.NF.A.1 CCSS.MATH.CONTENT.4.NF.A.2

Part 1: Shapes
Cut out rectangles (Appendix C) and shade in a different portion of each. Display the first and have students choose and record a fraction they believe a good estimate of the shaded amount shown. Have students share and then explain their answers with an accompanying whole class "thumbs up" to see who may agree. If students have trouble coming up with estimates, use the benchmarks: zero, the half and one whole. Listen to the ideas of many students to assess their fraction number sense and discuss why any particular estimate might be a good one. There is no single correct answer as long as estimates are in the "ball park". Display the shaded figure on the board and record the good estimate fraction below it where all can see. Repeat this process with additional shaded rectangles.

Part 2: The Number Line
Next, distribute copies of the number line to the class (Appendix C). Return to the displayed shaded figures and their accompanying good estimate fraction and have students place an "X" on their number line where they believe the fraction should go. Discuss and repeat.

Part 3 (Day 2)
Randomly, hand out fraction cards whose values are from zero to two and have students classify them on a classroom number line. When complete have students write to explain what fractions are equivalent, which fractions are the smallest, largest, etc.

Practice/Differentiation/Test
Give another set of fraction cards for students to classify on a number line. Have students include three written inequalities using each of the following symbols once: <, >, and =.

Adapted from Van de Walle
Activity 16.8, page 303

Activity 3

Fraction War – Comparing Fractions with Visual Models
Number of Players – 2
Duration: 45 minutes

Frayer	Essential Question(s)	Standard
pie fraction and other	How can I demonstrate	CCSS.MATH.CONTENT.4.NF.A.1

visual representation of fractions, improper fraction, mixed number	the value of a fraction using a visual model?	CCSS.MATH.CONTENT.4.NF.A.2

Print out several copies of circle fraction cards (Appendix D) and shade in to make fractions of varying values. Be sure these values include equivalent fractions.

Rules:
Shuffle the deck and deal the cards. Each player turns up a card at the same time. The person with the largest fraction takes all cards. If fractions are equal there is WAR. A WAR can progress in a number of ways by using one or more face down cards. The game is expedited the more cards students lay face down. Play continues until one player collects all cards.

Differentiation Ideas:
The player with the smallest fraction takes all cards.
Include other shapes or combine shapes to the visual fractions deck.
Add number fractions to the deck including improper fractions and mixed numbers.
Increase the numbers of players as you increase the size of the deck.

*Grade 4 expectations are limited to fractions with denominators of 2, 3, 4, 5, 6, 8, 10, 12, and 100.

Teacher can use this activity for small group intervention or the class can practice this skill as teacher works with his/her small intervention groups.

Adapted from Van de Walle
Activity 16.5, page 300

Bibliography

Aharoni, Ron, and Danna Reisner. Arithmetic for Parents: A Book for Grownups about Children's Mathematics. El Cerrito, California: Sumizdat, 2006. 132.
The author and research mathematician of this book is not far removed from today's classroom. He writes about the deep and abstract reasoning that is an integral part of the mathematics taught in 21st century elementary schools. A large section of the book serves as a guide for teaching primary and intermediate mathematics even by today's Common Core standards. His anecdotal stories, historical math factoids and ideas for teaching this content area are a tremendous guide and worth reading.

Flores, Alfinio, Erin Turner, and Renee Bachman. "Posing Problems to Develop Conceptual Understanding: Two Teachers Make Sense of Division Fractions."

Teaching Children Mathematics, no. October 2005, 117-21. Accessed November 16, 2014.
Many current educators and colleagues teach mathematics via the rules without reasons method. That is how most of us were taught as elementary school students. In order to teach for conceptual understanding, two teachers collaborate to pose their own problems on how to divide fractions in order to later teach the same to their students. They discovered that posing several problems of varying degrees of difficulty is what helps students internalize necessary concepts.

Hiebert, James. "Children's Mathematics Learning: The Struggle to Link Form and Understanding." The Elementary School Journal 84, no. 5, 497-513.
A distinction is made between form and understanding and the relationship between the two. Knowledge of form includes the rules, procedures and algorithms for problem solving. Understanding, on the other hand, is a student's intuition on how math works. This is evident, for example, as students demonstrate fair shares by dividing a candy bar in two pieces. When form and understanding are separated problem solving becomes mechanical and meaningless. Recommendations are made to keep both linked together in our teaching of mathematics.

"Mathematics Standards." Home. Accessed November 16, 2014. http://www.corestandards.org/Math/.
Many of the nation's math textbooks are weak on conceptual understanding, yet strong in rote and drill. As a result, many American school children do not possess the deep understanding that makes them successful in real life problem solving. The Common Core standards in math serve to help educators re-introduce this kind of rigor and align their teaching to help remedy this situation. Essentially all 50 states across the nation have adopted the Common Core math curriculum. One of its many assets is having a child from any grade level from any state sit in a classroom in the same grade level in another state and be taught the same content.

Walle, John A. "Teaching Through Problem Solving." In Elementary and Middle School Mathematics: Teaching Developmentally. 6th ed. Boston: Pearson /Allyn and Bacon, 2007.
The author creates a deep sense of the relevance for a problematic classroom. In my reflection of the reading, I am convinced of the great importance to create math problems for students to solve on their own and have begun to do so. When the teacher's role becomes facilitator, students are given opportunity to work on teams and demonstrate their potential for finding, testing and proving solutions.

Willingham, Daniel. "Is It True That Some People Just Can't Do Math?" American Educator, no. Winter 2009-2010, 14-19, 39.
This is the mantra of a certain group of math students in the classroom each year, yet an inaccurate conclusion by the same. As teachers of mathematics, it is essential to

eliminate any holes in student learning. Much simpler problems in math are embedded in the more difficult. For example, subtraction is embedded in long division and an understanding of algebraic equations depends on a conceptual understanding of the equal sign. Math takes time to learn and conceptual understanding builds upon conceptual learning. Conceptual understanding can produce problem solvers out of today's students.

Appendix A

Understand decimal notation for fractions, and compare decimal fractions:

CCSS.MATH.CONTENT.4.NF.C.5
Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. For example, express 3/10 as 30/100, and add 3/10 + 4/100 = 34/100.

Extend understanding of fraction equivalence and ordering:

CCSS.MATH.CONTENT.4.NF.A.1
Explain why a fraction a/b is equivalent to a fraction (n × a)/(n × b) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

CCSS.MATH.CONTENT.4.NF.A.2
Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

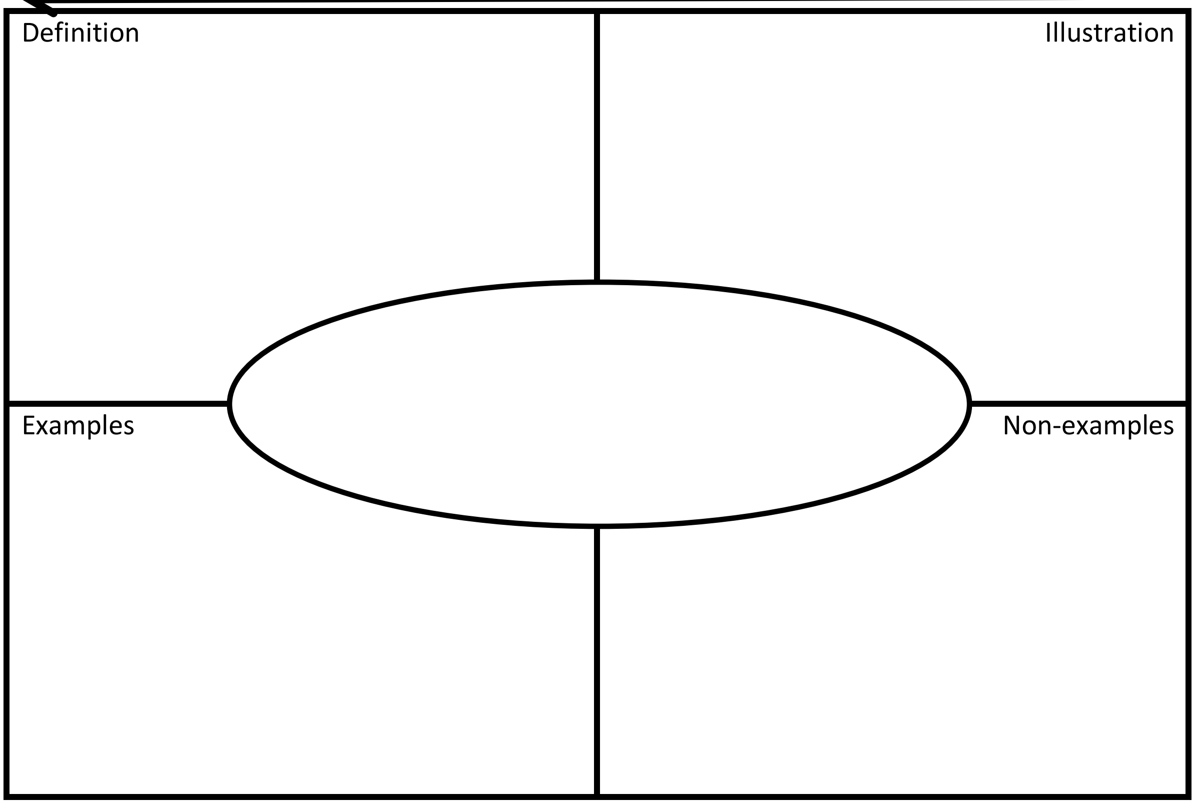
Appendix B

$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$
$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$
$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$

$\frac{10}{100}$	$\frac{20}{100}$	$\frac{30}{100}$
$\frac{40}{100}$	$\frac{50}{100}$	$\frac{60}{100}$
$\frac{70}{100}$	$\frac{80}{100}$	$\frac{90}{100}$

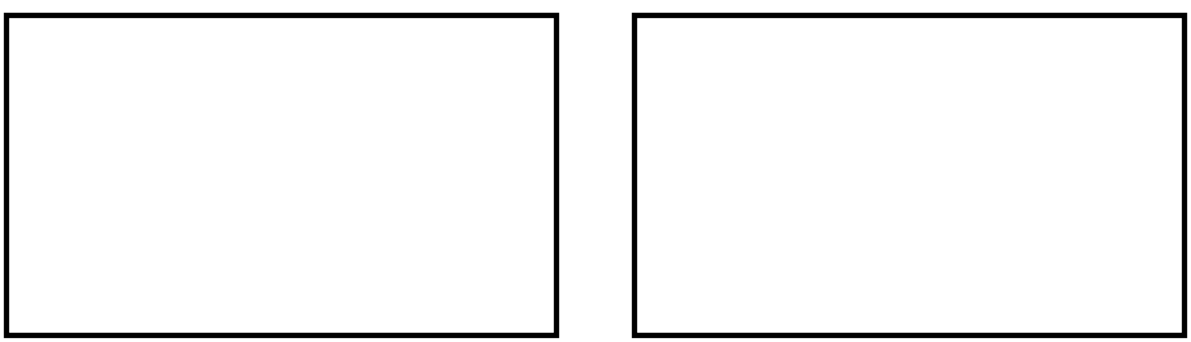
Appendix C

Frayer Model



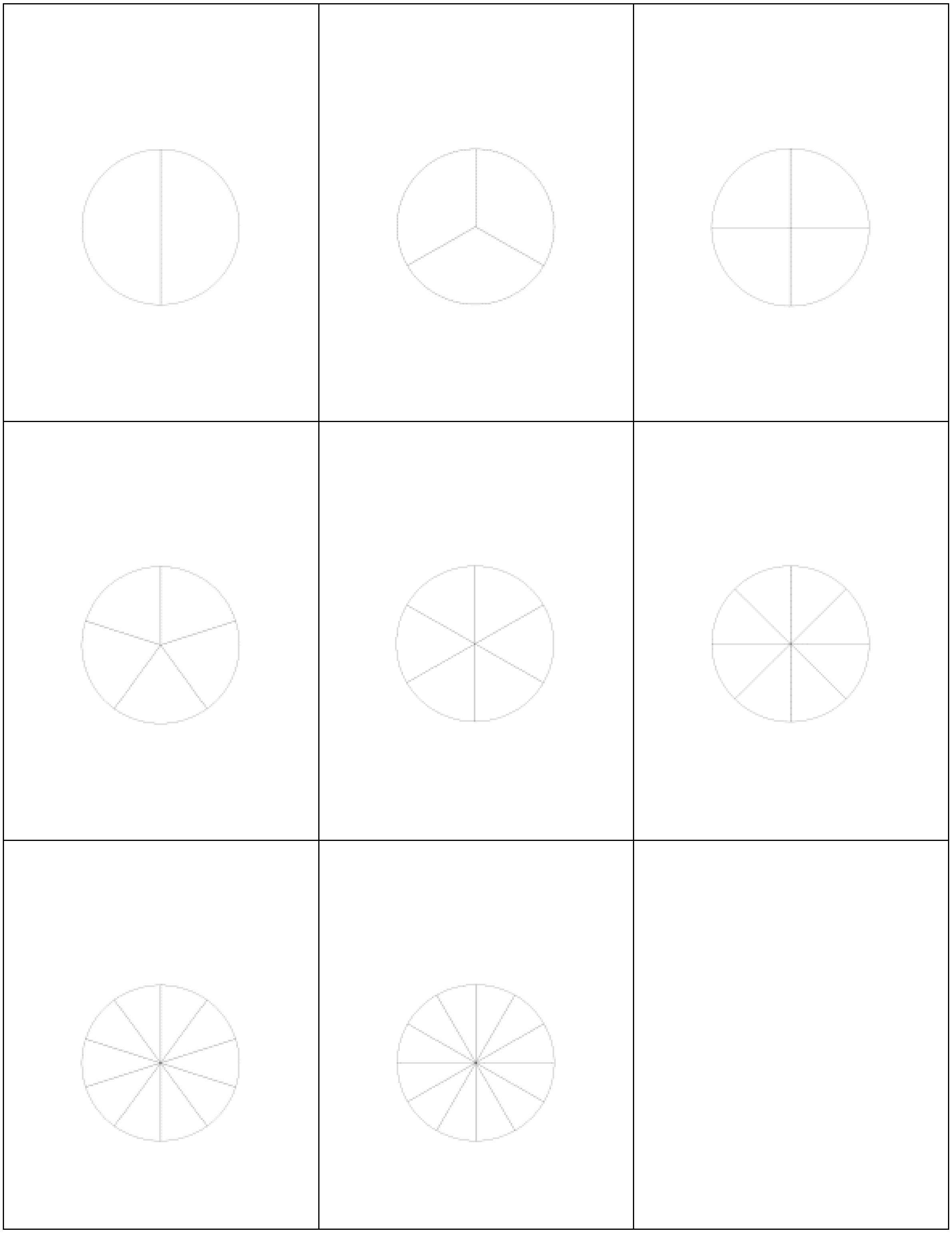
0 1 2

Number Line



Rectangles

Appendix D



Endnotes

1. “Mathematics Standards.” Home. Accessed November 16, 2014 http:wwwcorestandards.org/Math/.

2. James Hiebert, “Children’s Mathematical Learning: The struggle to Link Form and Understanding.” The Elementary School Journal 84, no. 5, 497-513.

3. Alfinio Flores, Erin Turner, and Renee Bachman. “Posing Problems to Develop Conceptual Understanding: Two Teachers Make Sense of Division Fractions”. Teaching Children Mathematic, no. October 2005, 117-21. Accessed November 16, 2014.

4. Daniel T. Willingham, “Is It True That Some People Just Can’t Do Math?” American Educator, no. Winter 2009-2010, 14-19, 39.

5. John A. Van De Walle, “Teaching Through Problem Solving.” In Elementary and Middle School Mathematics: Teaching Developmentally. 6th ed. Boston: Pearson / Allyn and Bacon, 2007. 294, 305, 303, 300.

6. Aharoni, Ron, and Danna Reisner. Arithmetic for Parents: A Book for Grownups about Children's Mathematics. El Cerrito, California: Sumizdat, 2006. 132.

Curriculum Unit Title: The Conceptual Understanding of Fractions

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KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Fractions are an important part of our numbering system and are used to represent numbers that are less than, equal to, or more than one whole.

ESSENTIAL QUESTION(S) for the UNIT

What kind of numbers are fractions and how are fractions used in the real world?

CONCEPT A	CONCEPT B	CONCEPT C
Comparing decimal fractions	Estimating visual representation of fractions & placing on a number line	Comparing multiple representations of fractions
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
How can I use models to show a fraction with a denominator of 10 is equivalent to a fraction with a denominator of 100? How do I explain in writing my understanding of comparing the numerator and denominator of fractions? Why should mathematical conjectures and their justification be recorded in	How can I use a number line to place and compare fractions?	How can I demonstrate the value of a fraction using a visual model?
VOCABULARY A	**VOCABULARY A**	**VOCABULARY A**
compare, equivalent, conjecture, justify	estimate, classify, compare, equivalent, number line	pie fraction(and any other shapes to visually represent fractions) improper fraction, mixed number

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES