Complex Numbers and the Imaginary Unit

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Introduction

Last year was the first time I had ever taught complex numbers. My curriculum has a unit on falling objects and, necessarily, my students and I investigate ways to find zeros of quadratic functions. This led us to trying to find the values of x for which the function $y = x^2 + 1$ is zero. From this foundational base, students would learn about the imaginary number *i*, complex numbers involving real and imaginary elements, how complex numbers are the solution to certain quadratic functions, the cycling powers of *i*, and would reach the summit of multiplying two complex numbers. Or, at least, students were supposed to learn those things. What I found were students not really grasping the challenging topic, but rather treated *i* as variable that was separate from all other numbers and not really understanding what it is.

My unit for the Delaware Teacher Institute is a 12^{th} grade unit on complex numbers and the imaginary number *i*. I would like students to have a sturdier understanding of what complex numbers are and some of their uses in mathematics. Furthermore, I want students to understand more about the addition and multiplication of complex numbers, and finally, experience a capstone activity of graphing complex numbers as vectors on the complex plane and represent the addition of complex numbers as vector addition.

School and Classroom Background

I teach at Thomas McKean High School in Wilmington, Delaware. Approximately two thirds of the students at my school qualify for free or reduced lunch. McKean is comprised of roughly 35% white students, 34% Hispanic students, 27% African American students, and 4% of other races. Additionally, about one fifth of our population receives special education services, which is higher than most high schools in our area. The course I teach that this unit will accompany is called Algebra with Statistics 4. I teach mostly seniors, with a sprinkling of higher-level juniors as well. The sections that I teach are at the college preparatory level, interspersed with students who elected to take the class for honors credit, who are given supplemental assignments to delve further into the content.

We use the fourth textbook from the series Interactive Mathematics Program. The first unit of this book, The Diver Returns, focuses on a central problem of a diver who intends to dive from a turning Ferris wheel into a tub of water that is on a cart that will pass beneath the Ferris wheel on a track on the ground. The goal is to give the time that the diver should leave the spinning wheel so that he lands into the water as the cart passes by. In this unit, students learn more about the circular functions sine and cosine, vectors, velocity, and falling objects. It is through exploring falling objects by finding out what input values of quadratic functions result in an output of zero that our curriculum extends into the complex realm[1].

Therefore, students enter my unit on complex numbers with prior knowledge of using the quadratic formula to find the zeros of quadratic functions as a means to determine, in context, what time a falling object will reach a given height. Additionally, my students have the lingering ghost of knowledge of how to multiply binomials, a skill that was visited two courses ago and hasn't been revisited since. Moreover, most students also have an understanding of exponents as a way to represent repeated multiplication.

My students lack prerequisite knowledge of graphing vectors and visually representing vector addition on a coordinate plane. Also, my students have a very weak understanding of set theory, as no effort in their past has been made to be explicit about sets of numbers. Both of these topics must be addressed in an intentional way within my unit so that my unit goals can be attained.

Objectives

I have seven objectives for my unit. The first is that I want students to understand what complex numbers are. That is, I want students to know that if we imagine that if we lift the rule that they are not allowed to take the square root of negative numbers, then we have a new set of more complex numbers with which to do math, the building block of which is the new number *i*, with the property that $i^2 = -1$. My second goal is that I want students to be able to add complex numbers and obtain a new complex number. Similarly, my third goal is that I want students to be able to multiply complex numbers and obtain a new complex number. Fourthly, I want students to analyze powers of *i* and be able to simplify the imaginary unit raised to large powers (for instance to recognize that $i^{3059} = i^3$). The fifth goal of my unit is to have students see complex numbers as solutions to some quadratic functions and to find complex solutions to quadratic functions. Sixthly, I want students to be able to represent a complex number on the complex plan as a vector with real and imaginary components. Finally, I want students to represent the addition of complex numbers graphically as the addition of two vectors in the complex plane.

Connection to the Common Core

The state of Delaware uses the Common Core State Standards and so will my unit. The standards we will cover will correspond to the objectives that I laid out earlier. Therefore, my standards are the following:

Common Core State Standards Addressed in Unit

CCSS.Math.HSN.CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real.
CCSS.Math.HSN.CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
CCSS.Math.HSN.CN.4 Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent that number.
CCSS.Math.HSN.CN.5 Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of their representation for computation.
CCSS.Math.HSN.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

Common Core Standards for Mathematical Practice

In this unit, students will be working on many problems and thinking in various scenarios individually and with peers that will require them to perform some of the Common Core Standards for Mathematical Practice. When students are engaged in my unit they will exhibit the following Practices:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Content

In mathematics, complex numbers are numbers of the form $a + bi$, where a and b are any real numbers and i is the imaginary number with the property that $i^2 = -1$. The set of all complex numbers is traditionally denoted as $\mathbb{C}$. Using this definition, when b is 0, we get the complex numbers that are also real numbers. Thus real numbers are a subset of complex numbers. Consequently, rational numbers, integers, whole numbers, and natural numbers are also subsets of complex numbers, as they are subsets of the set of real numbers $\mathbb{R}$.

Complex numbers are useful in solving quadratic equations that have their vertices above the x-axis and are concave up or have vertices below the x-axis and are concave down. When quadratics never cross the line $y = 0$ (the x-axis), complex numbers can be utilized to discuss the function's roots[2]. To determine the complex numbers that are the roots of a quadratic that never crosses the x-axis, we can use the quadratic formula. The resulting complex roots are always conjugates of each other: $a + bi$ and $a - bi$ (note: the a and b from the quadratic equation are almost certainly different than the a and b in the complex roots). In advanced mathematics and in sciences, complex numbers have many more applications, all of which are beyond the scope of my course and this unit.

Solving Quadratic Equations with Real Coefficients and Complex Solutions

Use the quadratic formula and express the solutions as conjugate complex numbers. Identify their real and imaginary parts. One can also reverse this process and find the quadratic function with real coefficients given its complex roots[3,4].

Operations with Complex Numbers and Algebraic Structures

If $z = a + bi$ is a complex number, then a is called the real part, and b is called the imaginary part. The complex number operations of addition and multiplication are defined as follows. When two complex numbers are added, the procedure is to add the real parts together and to add the imaginary parts together. So if we add complex numbers $(a + bi) + (c + di)$, the result would be $(a + c) + (b + d)i$, with $a + c$ representing the real component and $b + d$ representing the imaginary component of a new complex number which represents the sum of the given complex numbers. With multiplication, we treat the process in a similar way to multiplying two binomials. So, $(a + bi) \cdot (c + di) = (a \cdot c) + (a \cdot d)i + (b \cdot c)i + (b \cdot d)i^2$. Because $i^2 = -1$, this simplifies to $[(a \cdot c) - (b \cdot d)] + [(a \cdot d) + (b \cdot d)]i$, with $[(a \cdot c) - (b \cdot d)]$ being the real component and $[(a \cdot d) + (b \cdot d)]i$ being the imaginary component of the complex number representing the product of the two given numbers.

1. The properties of the operation of addition of complex numbers

The set $\mathbb{C}$ with respect to the operation of addition $(\mathbb{C}, +)$ has an algebraic structure of an abelian group. This is because the operation of addition has the properties of closure, associativity, identity, inverse, and commutativity.

Closure

If z and w are any two elements in $\mathbb{C}$, then $z + w$ is also in $\mathbb{C}$. So, with addition of complex numbers, if any two complex numbers are added together, the result will be a complex number.

Associativity

If z, w, and u are any three elements in ℂ, then (z + w) + u = z + (w + u). That is, when multiple complex numbers are added together, the way in which they are grouped together will not alter the sum.

Identity Element

There exists an element e in ℂ such that for any element z in ℂ, e + z = z + e = z. In the addition of complex numbers, the identity element is the number 0 which can be written as a complex number 0 + 0*i*. And for any z in ℂ, 0 + z = z + 0 = z.

Inverse Element

For any element z in ℂ, there exists an element z' in ℂ such that z + z' = z' + z = 0 (the identity element). For the addition of any complex numbers, the inverse element for a + b*i* is going to be –a – b*i*.

Commutativity

If z and w are any two elements in ℂ, then z + w = w + z. So, with addition of complex numbers, the order in which they are added will not alter the sum.

2. The properties of the operation of multiplication of complex numbers

The set ℂ with respect to multiplication (ℂ,·) has an algebraic structure of a monoid. This is because the multiplication operation has the properties of being closed, associative, and has an identity.

Closure

If z and w are any two elements in ℂ, then z · w is also in ℂ. So, the product of any two complex numbers will also be a complex number.

Associativity

If z, w, and u, are any three elements in ℂ, then (z · w) · u = z · (w · u). Thus, the order in which any three complex numbers are multiplied will not alter the product.

Identity Element

There exists an element e in $\mathbb{C}$ such that for any element z in $\mathbb{C}$, $e \cdot z = z \cdot e = z$. In the multiplication of complex numbers, the identity element is the number $1 = 1 + 0i$, and for any z in $\mathbb{C}$, $1 \cdot z = z \cdot 1 = z$.

If we remove the number zero from the set $\mathbb{C}$, then $\mathbb{C}\backslash\{0\}$ with respect to multiplication becomes an abelian group. It maintains the properties from before, when zero was still in our group, but gains the properties of inverse and commutativity.

Inverse element

For any non-zero element z in $\mathbb{C}$, there exists an element z' in $\mathbb{C}$ such that $z \cdot z' = z' \cdot z = 1$ (the identity element). For any complex number $a + bi$, its multiplicative inverse is the complex number $\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$.

Commutativity

If z and w are any two elements in $\mathbb{C}$, then $z \cdot w = w \cdot z$. That is, for any two complex numbers being multiplied, the order will not impact the resulting product.

Furthermore, we can look at how the set of complex numbers $\mathbb{C}$ behaves with respect to both addition and multiplication combined. Here, for $(\mathbb{C}, +, \cdot)$, we have a field structure. This is because $(\mathbb{C}, +)$ is an abelian group, $(\mathbb{C}\backslash\{0\}, \cdot)$ is an abelian group), and there is distributivity of multiplication over addition.

Distributivity

If z, w, and u are any three elements of $\mathbb{C}$, then $z \cdot (w + u) = z \cdot w + z \cdot u$. In short, multiplying a complex number by the sum of two complex numbers is equivalent to multiplying the first complex number by the other two individually and then adding the products.

Powers of *i*

Let's talk for a moment about the cycling powers of *i*. We know $i^1 = \sqrt{-1}$ and that $i^2 = i \cdot i = -1$. We can also look at $i^3 = i^2 \cdot i = -1 \cdot i = -i$. Similarly, $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = +1$. It is necessary, then, that every natural number power of *i* is either *i*, -1, -*i*, or +1. For instance, we can write i^{15} as $i^4 \cdot i^4 \cdot i^4 \cdot i^3 = 1 \cdot 1 \cdot 1 \cdot -i = -i$. This trick is achievable for any power of *i* that is a natural number.

Graphical representations of complex numbers

Complex numbers can be represented graphically on the complex plane. This is a plane with a horizontal axis of all real numbers (with respect to which the real parts of complex numbers are represented) and a vertical axis of all real numbers (with respect to which the imaginary parts of complex numbers are represented). With our axes constructed in this fashion, we can represent complex numbers a + b*i* as points in the plane with their coordinates given by the pair of coordinates (a, b), where the x-coordinate a is the real part of the number, and the y-coordinate b is the imaginary part of the number, and as vectors in the plane starting point at the origin and the endpoint at the point of coordinates (a, b).

Addition of complex numbers can be graphically represented through vector addition. This process requires us to start by constructing our first vector (representing the first complex number addend) and then constructing our second vector (representing the second complex number addend) starting at the "head" of the previous vector and treating that location like the origin of the axes. This "head to tail" addition method shows us where the resulting vector would be. It would start at the origin and end at the "head" of the second vector. The horizontal and vertical components of the resulting vector would correspond to the real and imaginary parts of the complex number. Iteration of complex numbers in the complex plane can result in interesting patterns and designs[5].

Activities

Activity #1

This activity is the beginning of my students' introduction to imaginary and complex numbers. This activity begins with having students find a solution to a basic quadratic equation with no real solutions, an act that they will believe impossible. It is then that students are shown the imaginary unit and given some details about it. Now, they are given square roots of negative numbers that they are expected to convert using the new imaginary number *i*. Students are next asked to solve other basic quadratic functions using *i*. Finally, students analyze the powers of *i* to discover their cyclical nature and create a shortcut to calculating i^n for any natural number n.

Imagine a Solution[6]

Find the x-values that make the equation $x^2 + 1 = 0$ true.

You find yourself having to take the square root of a negative number, something that you have been told is impossible. Well, what if we invent a new number that is the solution to this equation? That's exactly what mathematicians did several centuries ago, and it's time that you are introduced to their idea.

The number they created is represented by the symbol i and has the property that $i^2 = -1$, so $i = \sqrt{-1}$. This leads us to a whole new family of numbers like $3i$ and $-i$. These numbers are called **imaginary numbers** (numbers on the number line like 4, 0, -2, 7.5, $\sqrt{3}$, and π are called **real numbers**).

In this activity, we are going to perform tasks which utilize this new number i.

1. Write each square root in terms of this new number i. Explain your reasoning.

a. $\sqrt{-36}$

b. $\sqrt{-121}$

2. Use your reasoning from Question 1 to find solutions to each equation.

a. $x^2 + 9 = 0$

b. $x^2 + 62 = 13$

c. $x^2 - 10 = -15$

3. a. Investigate what happens when you raise i to different powers. For example, find the value of i^3, i^4, i^5, and so on.

b. Use your answer to find a simpler form of i^{3057}. Explain your answer.

c. Write a general procedure for finding the value of i^n in simple form without doing lots of repetition.

Activity #2

This activity is the introduction to complex numbers. It begins with a description of some vocabulary and definitions pertaining to complex numbers and progresses to adding and multiplying complex numbers. I was intentional about including a sum that will result in zero and the product of a pair of conjugates to allow students to see the special results that occur in those cases. Finally, the activity concludes with proving that given complex numbers are solutions to given quadratic equations and finding the complex solutions to other quadratic equations.

Complex Numbers and Quadratic Equations[7]

The imaginary numbers i and $-i$ are the solutions to the quadratic equation $x^2 + 1 = 0$ and $3i$ and $-3i$ are the solutions to the equation $x^2 + 9 = 0$. There are even imaginary numbers like $\sqrt{2} \cdot i$, which is a solution to the equation $x^2 + 2 = 0$.

Imaginary numbers provide solutions to many quadratic equations that have no real number solutions. However, some quadratic equations require a combination of a real and an imaginary number. These combinations are called **complex numbers**. All complex numbers can be expressed as the sum of a real number and an imaginary number, such as $3 + 2i$ and $-4 - 9i$ (which is equivalent to $-4 + -9i$). Complex numbers reveal themselves as solutions to many quadratic equations.

The general form of a complex number is $a + bi$. The a term is called the **real part** and the term bi is called the **imaginary part**. Either part of a complex number could be zero.

With this new set of numbers, it's important to establish the rules of arithmetic of complex numbers. So let's try some addition and multiplication!

1. $(4 + 8i) + (-2 + 3i)$

2. $(-14 - i) + (14 + i)$

3. $(4 + 8i) \cdot (-2 + 3i)$

4. $(14 - i) \cdot (14 + i)$

The final part of this activity is centered on the simplest complex number that has both nonzero real and imaginary parts: $1 + i$.

1. Show that $1 + i$ is a solution to the equation $x^2 - 2x + 2 = 0$.

2. Apply the quadratic formula to the equation $x^2 - 2x + 2 = 0$ to find a second solution.

3. Verify that the second solution satisfies the equation.

4. Use the quadratic formula to find the solutions to the equation $x^2 - 4x + 7 = 0$. Check both solutions.

Activity #3

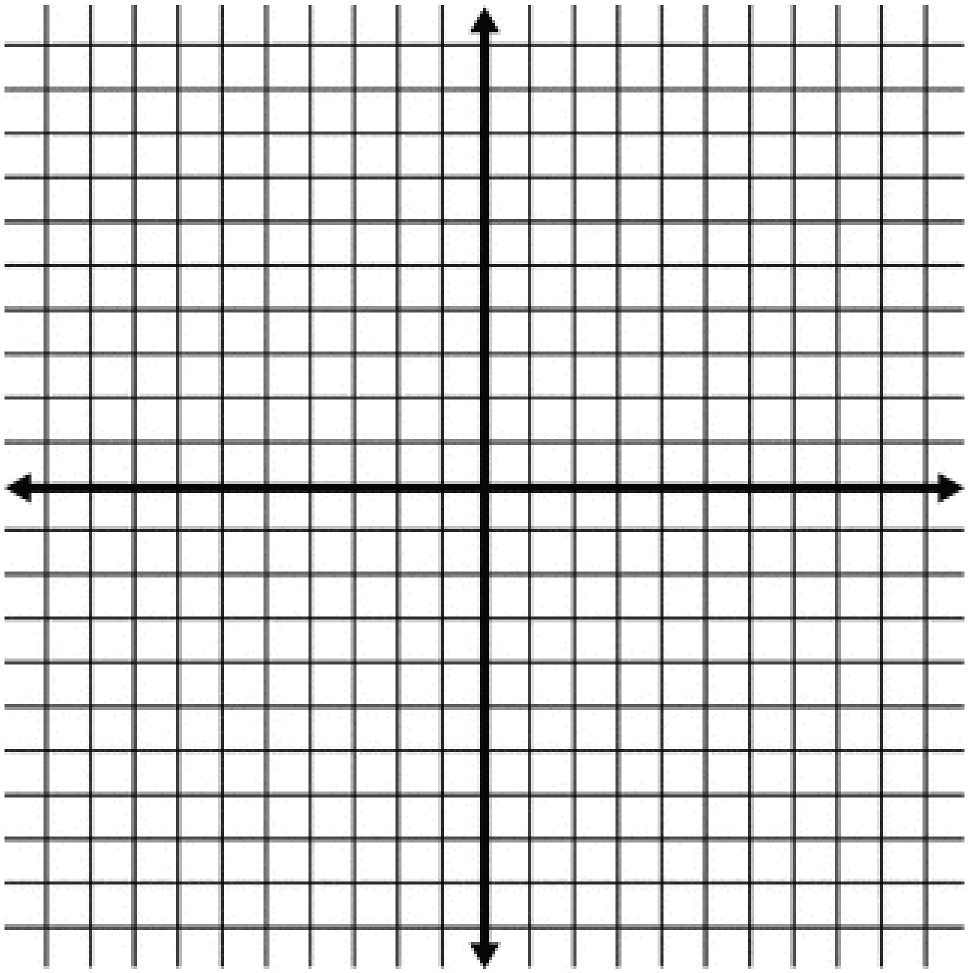
This activity is the capstone activity of my unit. In this activity, students view the addition of complex numbers through the lens of graphing the addition of vectors. It begins with an introduction to how to graphically represent complex numbers on the complex plane and has students graph a few complex numbers as practice. The activity then moves to a review of addition of complex numbers and has students practice that addition. Finally, it has students graph vectors that represent the complex numbers that served as the addends earlier and graph them head to tail and then graph the sum vector.

Graphing Complex Numbers[8]

You have seen that when complex numbers are added together, mathematicians add the real components and the imaginary components separately. So $(a + bi) + (c + di) = (a + c) + (b + d)i$. In this activity, you will graph complex numbers and the addition of complex numbers. When we graph complex numbers, we treat them like vectors. The real component is represented on the horizontal axis and the imaginary component is represented on the vertical axis. For the complex number $a + bi$, we construct the complex vector from the origin to the point (a,b).

On the axes below, graph the following complex number vectors:

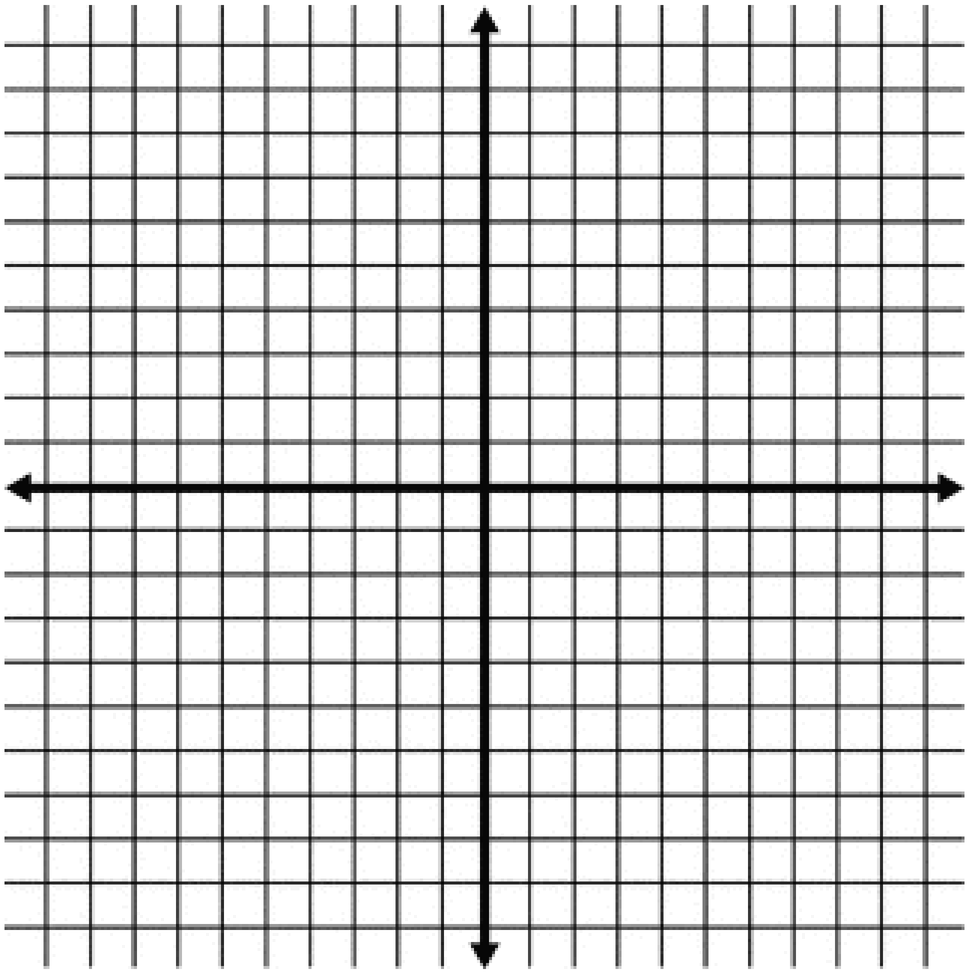
a. $3 + 4i$ b. $-7 + 2i$ c. $5 - i$ d. $-8 - 3i$ e. 2 f. $5i$



Now calculate the sum of these complex numbers:

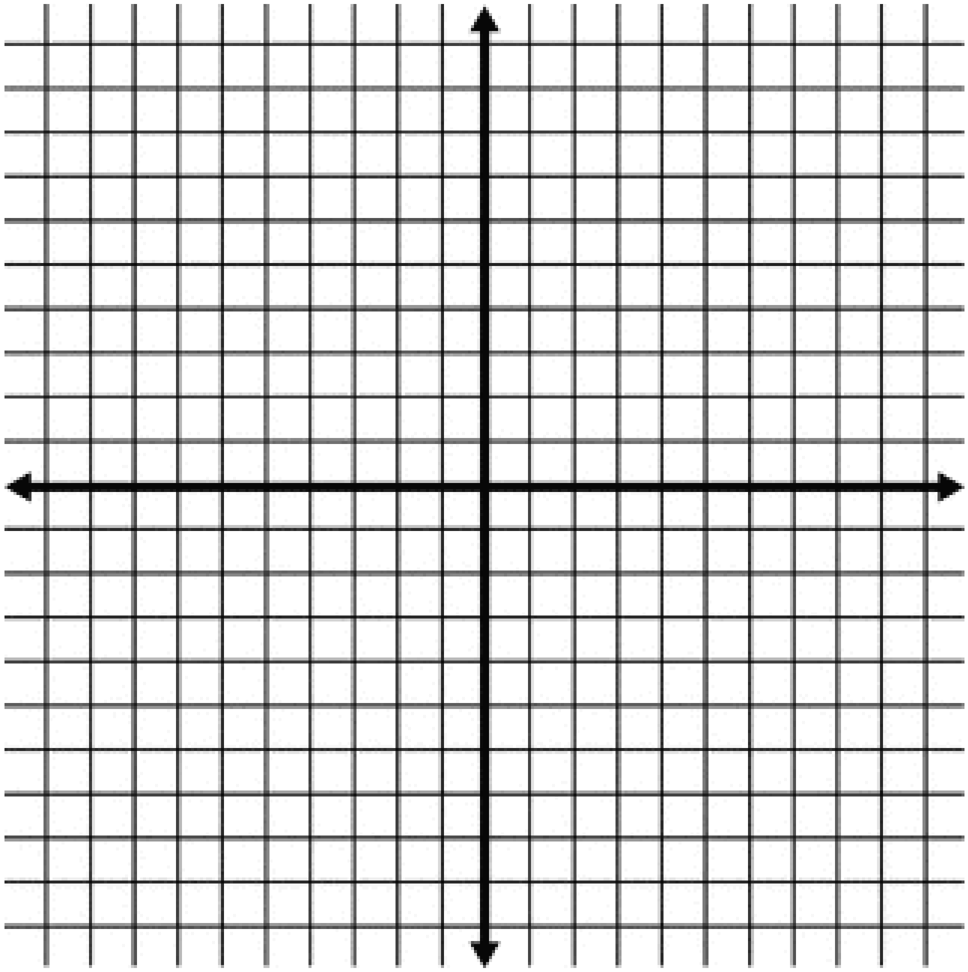
1. $(3 + 4i) + (2 + 2i) =$
2. $(-5 + i) + (7 – 3i) =$
3. $(1 + 3i) + (2i) =$
4. $(8) + (-2) =$

Now graph each of these complex numbers (the two addends from each problem **and** the sum you found) on the complex plane below, with one caveat: graph the first addend and the sum starting from the origin and graph the second addend as if it began at the end of the first addend vector.



Summarize your findings:

Now graph each of these complex numbers (the two addends from each problem **and** the sum you found) on the complex plane below, with *this* caveat: graph the **second** addend and the sum starting from the origin and graph the **first** addend as if it began at the end of the second addend vector.



What do your finding reveal about the addition of vectors?

Conclusion

These activities are meant to improve my students' understanding of imaginary and complex numbers, the cycling powers of *i*, the process of arithmetic of complex numbers, complex numbers as solutions to quadratic equations, and visual representations to complex numbers and the sums of complex numbers. In the past, I have seen students struggle with conceptual understanding of complex numbers, the imaginary number *i*, and set theory in general. It is my hope that through exposure to this unit and a teacher who has a sturdier background in this content as a result of this unit, my students and other students will be more successful in their complex number endeavors.

Bibliography

Fendel, Dan, Diane Resek, Lynne Alper, and Sherry Fraser. "The Diver Returns." In *Interactive Mathematics Program Year 4*, 25-28. 2nd ed. Emeryville, CA: Key Curriculum Press, 2012.

This is the textbook for my course. Within this text lies the activities that I revamped and added to in creating my activities.

McNamara, Meghan E. "Complex Conjugates and Zeros of Polynomials." *The Mathematics Teacher* 99, no. 7 (2006): 469.

This article compares two processes of finding a quadratic equation given its complex roots.

Metz, James. "Quadratics I: Finding Complex Roots on the Real Plane." *The Mathematics Teacher* 104, no. 3 (2010): 170-71.

This article gives a great way to conceptually visualize the complex roots on the real plane and describes why the complex root is truly a root of the function.

O'Dell, Robin S. "Not So Complex: Iteration in the Complex Plane." *The Mathematics Teacher* 107, no. 8 (2014): 592-99.

This article presents an extension of complex numbers. It is revelaed how iterations of complex numbers graphically can create interesting patterns and designs.

Schuloff, Anita. "Complex Conjugates." *The Mathematics Teacher* 100, no. 6 (2007): 391.

This article provides another method for finding a quadratic equation given one of its complex roots.

Notes

[1](Fendel, Resek, Alper, and Fraser, 2012) p. 25-28
[2](Metz, 2010) p. 170-171
[3](McNamara, 2006) p. 469
[4](Schuloff, 2007) p. 391
[5](O'Dell, 2014) p. 592-599
[6](Fendel, Resek, Alper, and Fraser, 2012) p. 25-26
[7](Fendel, Resek, Alper, and Fraser, 2012) p. 27
[8](Fendel, Resek, Alper, and Fraser, 2012) p. 28

Curriculum Unit Title

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KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent that number. Solve quadratic equations with real coefficients that have complex solutions.

ESSENTIAL QUESTION(S) for the UNIT

How would you add two complex numbers?
How would you multiply two complex numbers?
How would you know if a complex number is a solution to a quadratic equation?
How can you represent a complex number graphically?
What do you predict the addition of Complex Number Vectors to look like on our graph?

CONCEPT A	CONCEPT B	CONCEPT C
Arithmetic of Complex Numbers	Complex Numbers as Solutions to Quadratic Equations	Complex Numbers on the Complex Plane
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
How would you add two complex numbers? How would you multiply two complex numbers?	How would you know if a complex number is a solution to a quadratic equation?	How can you represent a complex number graphically? What do you predict the addition of Complex Number Vectors to look like on our graph?
VOCABULARY A	**VOCABULARY B**	**VOCABULARY C**
Complex Number Real Part Imaginary Part	Conjugates	Complex Vector Vector Sum

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

IMP4 Textbook

Calculators

Rulers