Moving Third Graders From Guessing and Checking to Reasoning and Proof for Understanding Multiplication

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Introduction

When I was in elementary school we had to memorize our multiplications facts. Mrs. Carney would have us stand up in the front of the room and recite the facts through twelve. Each week we were expected to memorize another row in the multiplication table and after twelve weeks, we were expected to be able to recall any multiplication equation up through the twelve's table. I admit, I am pretty quick at my multiplication fact recall and amaze my third graders each year as they cannot stump me on any fact. But as I prepare lessons for my students, I can't help but wonder how much easier middle school math and high school math would have been if I had been building a conceptual understanding rather than just memorizing facts. I do believe that automaticity of the facts will aid my students when they are searching for a solution and one goal is for them to build this skill.

However, fluency is not the only goal I have for my students. I would like for my students to understand the concept of multiplication and be able to reason about a problem, test a conjecture, and justify their answers. One of the problems solving strategies we have used in my classroom is called "Guess and Check". This strategy requires a student to guess a solution, test its correctness, and change the guess using logical reasoning. I believe the terminology of "guess and check" over simplifies the process. Students at this early stage see the word "guess" and make a blind grab of any number and try to make it work. They see the word "check" and more often than not take this to mean that they need to look to see if they have an answer. They lack the awareness that, at this step, they should be justifying their solution. My objective is to change the terminology in my classroom to send them on the path to better understanding. I want my students to combine given information with mathematical intuition to make a reasoned guess when prompted.

Demographics

The Red Clay Consolidated School District is located in Northern New Castle County, Delaware with a combination of urban and suburban settings. Some of its elementary schools are located in the heart of the largest city in the state. The district is comprised of 29 schools with approximately 1000 teachers. It services over 16,000 students. Of those students, 27% are African American, 4% are Asian, 20% are Hispanic, and 49% are White. Students' needs vary, with almost 15% receiving Special Education Services and 10% receiving English Language support. In addition, 41% of the students come from families with low incomes.

Highlands Elementary is an urban school in the city of Wilmington, Delaware. We are a small K-5 school with an enrollment average of 300 students. Our minority population represents 86% of our student body with 87% of the students falling into the low socio-economic status. Recently we were named a Priority School by the state Department of Education. With this designation in place, our school needed to develop a turnaround plan with specific programs, strategies, and interventions to improve students' test scores. One of our interventions includes increasing instructional time for Mathematics. All students will receive 90 minutes of Mathematics Instruction daily. This is adding thirty minutes daily to our mathematics instruction. With this additional half hour, students will participate in math discussions, work in small group, use technology to enhance understanding, in conjunction with a new math series the district is adopting.

My classroom is a third grade classroom usually comprised of 20-25 students. I have a mixture of abilities with approximately 20% of my students receiving special education services. A Special Education teacher pushes into my classroom for one half hour during Math instruction.

Background Information

Reasoning

Mathematical reasoning and proof can be developed in students at a young age. As they look for structure and patterns in the lives around them, they begin to ask questions that try to discover what effect these patterns have on them. Why do we always stop at a red light, what would happen if we did not? When I am crossing the monkey bars at recess, why is it easier for me to grab the next bar when I swing my whole body forward? If I count by two's, am I always doubling the number? Why? Students who reason in their everyday lives can bring this skill into the math classroom as they look for these same types of patterns in the manipulations of numbers. In fact, developing reasoning in my students, in my opinion, is essential for them to understand math. Reasoning strategies will assist them while exploring topics, justifying their solutions and the solutions of others, and determining which solution pathway makes sense. Building the skill of reasoning will not only improve my students' understanding of math, but hopefully will also become a habit that will affect every facet of their life.

Students will develop and apply reasoning skills to make mathematical conjectures, plan and construct organized mathematical arguments, assess these conjectures and justify their conclusions Students make sense of mathematics through reasoning. Reasoning can be defined as an organized, analytical, well-reasoned approach to learning mathematical concepts and processes and to solving problems which requires an emphasis on reasoning. Even at an early age, students can begin to develop reasoning skills that will aid them in making sense of new ideas and to cultivate habits beneficial to their mathematical learning.

My third grade students can be asked to make a rule about the difference between even and odd numbers. As they begin their investigations, they make a conjecture, even numbers have no leftovers. They then will create a plan to test this conjecture, let's use counters of varying amounts and put them into two equal groups. They then will develop a "rule" that they would be able to use to describe how to determine the oddness or eveness of any number. Our next step would be to reason about adding two numbers together. What will the sum be if we added two even numbers together? What would the sum be if we added two odd numbers together? What would the sum be if we added one even number and one odd number together? Students exploring with manipulatives, patterns, and procedures will build these reasoning skills that I am searching for within my students. The next step in the building process would move to the multiplication operation, keeping the investigation process the same and have students continuing to investigate the outcomes and develop an understanding. Through this process, we will hopefully avoid an overgeneralization of understanding through one example. Students will be able to see for themselves that 9 is odd because there is a leftover counter, will they see however that when you add one odd number with a leftover to another odd number with a leftover, that these two leftovers will form another unit, thus yielding an even result? With manipulation of materials and reasoning skills, I hope so!

Proof

A proof is a mathematical argument, using precise language, definitions, and reasoning to validate their conclusion. The trick here will be to work within the mental understandings of third graders with regards to expectations of the proofs they produce. My overall goal is for my students to provide some form of justification for their answer and the following three proofs will be introduced through this unit.

Direct Proof

"A direct proof is one of the most familiar forms of proof. We use it to prove statements of the form "if p then q" or" p implies q" which we can write as $p \Rightarrow q$. The method of the proof is to take an original statement p, which we assume to be true, and use it to show directly that another statement q is true."[1]

"During our study of proofs, we will use several approaches. The most straight forward, called direct proof, proves the a proposition in the form If P, then Q by assuming P (called the hypothesis) is true and, through a sequence of logical deductions, shows that Q (the conclusion) must be true. Some hints: Be sure you are convinced the proposition you are trying to prove seems true. Try some examples and look for patterns you can exploit. Be sure you know what the conclusion should be. Think of proofs as like doing a problem where you know what the answer should be – you are trying to work toward it. In a very real sense you're trying to build a bridge from the hypothesis to the conclusion."[2]

Proof by Exhaustion

"A proof by exhaustion is a proof by cases where every possibility (usually a small, finite number) is checked separately."[3] By posing the problem; "Mary has three different flavors of cupcakes; vanilla, chocolate, and strawberry and three different flavors of frosting; vanilla, chocolate, and strawberry. How many different combinations of cupcakes can Mary make?"; students can create a list and create every possible combination.

Proof by Contradiction

"To prove a statement P **by contradiction**, you assume the *negation* $\sim P$ of what you want to prove and try to derive a *contradiction* (usually a statement of the form $A \wedge \sim A$). Since a contradiction is always false, your assumption $\sim P$ must be false, so the original statement P must be true."[4]

"Key to all mathematics is the notion of proof. We wish to be able to say with absolute certainty that a property holds for all numbers or all cases, not just those we've tried, and not just because it sounds convincing or would be quite nice if it were so. Certain types of proof come up again and again in all areas of mathematics, one of which is proof by contradiction.

To prove something by contradiction, we assume that what we want to prove is not true, and then show that the consequences of this are not possible. That is, the consequences contradict either what we have just assumed, or something we already know to be true (or, indeed, both) - we call this a contradiction.

A simple example of this principle can be seen by considering Sally and her parking ticket. We know that if Sally did not pay her parking ticket, she would have got a nasty letter from the council. We also know that she did not get any nasty letters. Either she paid her parking ticket or she didn't, and if she didn't then, from our original information, we know that she would have got a nasty letter. Since she didn't get a nasty letter, she must therefore have paid her ticket.

If we were formally proving by contradiction that Sally had paid her ticket, we would assume that she did not pay her ticket and deduce that therefore she should have got a nasty letter from the council. However, we know her post was particularly pleasant this week, and contained no nasty letters whatsoever. This is a contradiction, and therefore our assumption is wrong. In this example it all seems a bit long winded to prove something so obvious, but in more complicated examples it is useful to state exactly what we are assuming and where our contradiction is found."[5]

Multiplication Problem Types

Multiplication is a skill that is an important building block for third graders. I introduce the concept by making connections to repeated addition. In multiplication, we are repeatedly adding a specific set of numbers. Through this process, students are looking for and recognizing patterns that develop. Students are shown that we have factors that generate a product. The factors represent group size and number of groups. The product represents the total number of objects. There are a variety of problem types involving multiplication. When students are exposed to the different types, they can begin to make sense of the problem and build their conceptual understanding. Students will solve multiplication equations unknown product, group size unknown, number of groups unknown. We investigate problems that model equal groups, arrays, area, and comparison, thus creating a matrix of sorts. I will provide examples of problems that will fit into the problem solving matrix.

Equal Groups- Unknown Product

Students will solve a variety of problems that give the students the group size and the number of groups in the set and they will need to solve for the product. There is an emphasis on the requirement that the group sizes remain equal. Brian has 4 shelves and on each shelf he has 5 books. How many books does he have in all? A student could write an equation of 4*5= b to solve this multiplication problem.

Group Size Unknown

Brian has 4 shelves and the same number of books on each shelf. If Brian has 20 books altogether, how many books are there on each shelf? A student would write an equation of 4*b=20. As we make connections in our learning, I would also begin to share how division is related to multiplication. Students could write an equation 20/4=b.

Number of Groups Unknown

Brian has some shelves and each shelf has 5 books on it. If Brian has 20 books altogether, how many shelves does he have? A student could write an equation of b*5=20. The division equation would be written as 20/5=b.

Arrays

Arrays are a great way to introduce models to multiplication. This is a systematic organization of objects into rows and columns. This strategy can be useful when investigating the Commutative Property. Students determine that they can use the rows as the multiplier or the columns and the product remains the same.

Unknown Product

Sarah has a picture album. She has 4 rows of pictures with 5 pictures in each row. How many pictures does Sarah have in all? A student could write the equations 4*5= p or 5*4=p to represent this problem.

Unknown Factors

Sarah has a picture album. She has 4 rows of pictures with the same number of pictures in each row. She has 20 pictures. How many pictures are there in each row? A student could write the equations 4*p= 20 or p*4=20 to represent this problem.

Sarah has a picture album. She has the pictures arranged in rows. There are 5 pictures in each row. She has 20 pictures. How many rows of pictures does she have? A student could write the equation p*5=20 to represent this problem.

Area Model

Students are frequently taught multiplication through the area model. I do use this model in my classroom and teach it as a strategy. Students can create the area grid and count the boxes as an entry point to modeling with mathematics and drawing pictures to justify their solutions. Students create a rectangle and can make rows and columns to coincide with the length and width. Using this model is also useful when investigating the Distributive Property of Mathematics

Unknown Product

We need a new rug for our reading area in our classroom. The space for the reading area will be 4 feet long and 5 feet wide. What will be the area of the rug? A student could write the equation 4*5=r to represent this problem.

Unknown Factors

We need a new rug for our reading area in our classroom. The area of the rug will be 20 feet. The space is 4 feet long, how wide is the space? A student could write the equation 4*r=20 to represent this problem.

We need a new rug for our reading area in our classroom. The area of the rug is 20 feet. The space is 5 feet wide, how long is the space? A student could write the equation r*5=20 to represent this problem.

Comparison

Unknown Product

Miss Gormley's class has read 4 times as many books as Mr. Slater's class. If Mr. Slater's class read 5 books; how many books did Miss Gormley's class read? A student could write the equation 4*5= b to represent this problem.

Unknown Factors

Miss Gormley's class has read 4 times as many books as Mr. Slater's class. Miss Gormley's class read 20 books. How many books did Mr. Slater's class read? A student could write this equation 4*b=20 to represent this problem.

Miss Gormley's class has read more books than Mr. Slater's class. Mr. Slater's class read 5 books and Miss Gormley's class read 20 books. How many times as more books did Miss Gormley's class read? A student could write this equation b*5=20 to represent this problem.

12 Things to do to improve your math instruction

As I have grown as a mathematics teacher, I have read numerous articles on effective mathematics instruction and engaging students. Through my research and participation in a variety of professional development opportunities I have discovered lists, suggestions, and tips for improving my instructional practices. The following list is a compilation of these thoughts married with my personal observations.

1. Set an expectation in your classroom that students should do what makes sense to them. If they are given specific strategies or algorithms that they have not internalized or understand, they will have difficulty making sense of problems in new situations. Allowing students to investigate and develop their own strategies, gives them the belief that they too can do it.

2. Have students explain their thinking and reasoning in all instances. Many of my students want to explain their solutions with a quick "I did it in my head" answer. I probe further and ask my students, "What did you do in your head?"

3. Encourage students to talk with one another during class to share strategies, solutions, and questions. This allows all students to develop learn strategies and solution paths from peers. Sometimes a students can summarize or explain a problem to another student better than I can explain it to them. Additionally, when a student is with just a partner, they sometimes are more willing to admit confusion than they would admit in front of the entire group.

4. Make writing an integral part of math learning, my students write in their math journals daily. We show solutions to problems but they also create ‘rules’ that they try to prove, they pose questions they still have, and they summarize their learning for the day.

5. Embed math activities in context. It continues to amaze me that a student may have difficulty in following a procedure for multiplication, yet if I construct a reason for the operation, their understanding increases. If Noel reads 10 pages each night for homework, how many pages will she read in one week? How many in a month? Also, I have found the little act of inserting student’s names into your problems surprisingly increases engagement and excitement.

6. Use manipulative materials whenever possible. Too many times teachers remove manipulative materials from students too soon. Try to look at these materials as a resource not a crutch and if they still use the resource, they still need it to build meaning. Some students may not need the manipulatives to build understanding, yet the use of them will enhance their understanding. “Manipulatives are important when you teach for the understanding of math concepts. They are the concrete objects you provide in order to transfer understanding to the abstract level. Don't place them on a desk or table and wait for a student to discover what to do with them. Demonstrate the manipulatives letting the students use them while you teach. Provide specific activities so your students know how to use the manipulatives by themselves. Manipulatives can be a valuable teaching tool if they are properly used.”[6]

7. Bring the quality and richness often apparent in students’ writing and art into their math work. Students hear that they need to add details to both their art and their writing pieces. Using this same requirement in the math classroom will enable students to appreciate the consistency across subject areas.

8. Make calculators available to children at all times. This statement leads to many discussions about the calculator being a crutch. In my experience, many students eventually stop using the calculator as they realize their knowledge is better than the calculator or its use slows them down. Some students need it to build confidence, some need it to build fluency, and some students will always need it based on their learning styles or needs. Teach your students to use it appropriately.

9. Let children push the curriculum rather than having the curriculum push the children. This statement will also raise some eyebrows as we all are aware of pacing

schedules and assessments schedules. Again think about the needs of your students and do the best you can to ensure they are building understanding.

10. Keep an eye out for instructional activities that are accessible for students with different levels of interest and experience. Not only keeping an eye out, don't be afraid to develop activities yourself. In this day of the internet when you can find millions of activities with the click of a button, try to develop the activities yourself. You know your students the best and can meet their needs. Additionally, you can never be certain that the activities you find will teach the concepts you are looking to teach in an effective, correct way!

11. Remember that confusion and partial understanding are natural to the learning process. A new term that I hear in mathematics is "productive struggle". This term describes what I envision what happens daily in my classroom. A few years ago I would see my students struggle and get frustrated and I would jump in with hints or suggestions. I notice now that I inadvertently was allowing my students to give up when the going got tough.

12. Take delight in your students' thinking. Nothing beats the "aha" moment that arrives when a student puts all the pieces together and understanding clicks into place. I enjoy listening to the pathway they take as they explain how they know they are correct.

Problem Solving Process

In my experience, when I talk about problem solving, many of my colleagues think I am speaking of word problems. I take a minute to explain to them the difference; word problems are math exercises that embed numeric equations into a variety of questions, and problem solving involves implicitly teaching students strategies to solve a variety of problems. There are a set of steps that students need to follow in order to become successful when they begin the problem solving process. As I work to develop the reasoning abilities of my students, I plan to use a large variety of problem solving opportunities. These problem solving opportunities also help my students develop the practice of making sense of a problem and persevering while solving it.

In my classroom, I have found that there are seven strategies that are appropriate and useful to my students: draw a picture, look for patterns, make a chart or graph, guess and check, work backwards, make a list, choose an operation. Each strategy is introduced along with several problems that lend themselves to that specific strategy. I also provide my students with a graphic organizer to help them organize and make sense of the problems. I am not a big fan of teaching key words because there are always a few problems that do not fit the key word rules and I think this also teaches students to focus on a set of words and not to think holistically of the problem.

Understand the Question: students need to read the questions carefully and develop an understanding of what the question is asking. Many misconceptions and errors began when students answer a different question than what was being asked.

Choose a Plan: as students begin to work with the problem, they need to decide which strategy will best aid them.

Try your Plan: this is the place in the problem solving process that students put their ideas into action. They are thinking about each step as they proceed and continue or make changes if necessary.

Check your Answer: Once students come to a solution they need to ensure their response is accurate. They should ask themselves some questions to guide their thinking. Did you answer the question that was asked? Does your answer make sense? Did you remember to use the correct units? Then they should redo the problem another way and try to get the same answer and check your math work for small errors. After the solution has been determined students should then, Reflect: Think about what you have done and what you have learned. Also, students should ask themselves if there is anything they are still confused about.

Understanding the Problem using a KWCSA

Using the Standards for Mathematical Practice as a guide, I have worked to develop strategies that aid my students as they make sense of problems and persevere in solving them. My students use a revised KWL form specifically adapted to help in my math classroom. We call the graphic organizer a KWCSA chart. The K section asks students; What do you KNOW about the problem? This enables students to clarify the information within the problem and provides them a place to record information they will need to solve the problem. They must also make decisions to justify what information is needed to solve the problem and what information is superfluous. The W section asks students, What do I WANT to find out? Many times my students get confused as to what they are actually being asked in the problem and this gives them a place to write it down and focus on what they are solving. The C section asks students; Are there any CONDITIONS, rules or tricks I need to look out for? The S section asks students to list two to three STRATEGIES that they believe will help them solve this problem. Multiple strategies are listed so students know if one strategy is not working they can try another. The A section is the place where students record their ANSWER. I was finding that many of my students would work hard to solve a problem and then never finalize their work. This space reminds them to refer back to the W section and make sure they have answered the question they were asked.

Connections

Having students make connections will improve their understanding and use of the correct terminology. Students can make a variety of connections; math-to-self, math-to-world, math-to-math. Math-to-Math connections encourage students to connect prior knowledge and experience to the current concept. They should ask themselves what does this problem remind me of? Have I ever worked on a problem similar to this one? Math-to-World connections have students thinking about events, the environment, and natural or created structures. Students could ask themselves, is this problem related to any other type of problem I have seen? Who would use this type of math? Have I seen anything like this on TV or in the movies? Math-to-Self connections help students use personal experiences to understand the problem. Students could ask themselves, Where would I use this problem? Is there a tricky part that I need to thing about? What do I already know?

Vocabulary

A good definition should be unambiguous and should allow students to test and check whether an object or situation fits the definition. In terms of multiplication, many teachers always ask, "does it matter which number you write first?" I believe if we keep the specific mathematical terms in mind and use them often during instruction, this matter will resolve itself. The first number in a multiplication equation should be the multiplicand and the second number in a multiplication equation is the multiplier. The solution to a multiplication is known as the product. I prefer to use the term product in my classroom; the term "answer" carries with it an implication that there should be one right solution. During vocabulary introduction, I will use strategies that are successful in other content areas in my classroom. Students will make a self-analysis based on their understanding and comfort with a word and place their information onto a class data table. Students will choose from the following options; "I would like to learn this word", "I have seen this word but I am not sure of its definition", and "I know this word and I can use it accurately". After each student adds their information to the table, we discuss the data set of the class and move forward with learning more about the term. The terms are posted on our Math Wall for students to refer to during the day. Using the appropriate terms reinforces the mathematical practice dealing with attending to precision.

Teaching Strategies

Small Groups and Centers

Small groups and centers allow me the opportunity to differentiate the learning process and provide remediation to some students and enrichment to others. I try to commit one day a week to this strategy as it allows students the chance to apply and practice skills. Centers should provide meaningful, independent work for the students and should be open-ended in order to provide students with multiple entry points and solutions. It is important to set up routines in order to build independence and insure engagement.

Gallery Walks

Students will work in groups of 3 or 4 and will be presented with a problem to solve. The problems should be written at the top of large poster paper. Place the posters around the room. Students work together to solve the problem. After a designated time period, depending on the difficulty of the problems I go for about 5 to 10 minutes, groups move to another poster. Students review the previous solution and then solve the problem using a different strategy. I usually allow for 3 rotations and then groups will present the problems to the class and a discussion will take place about the variety of strategies.

Journals

Students should be writing mathematically every day. The use of journals can take on many different purposes. The journal is a place for students to have a work space for problems. “While students learn how to "do" math, they must also learn how to articulate what they are learning. It is important to provide many opportunities for students to organize and record their work without the structure of a worksheet. Math journals support students' learning because, in order to get their ideas on paper, children must organize, clarify, and reflect on their thinking. Initially many students will need support and encouragement in order to communicate their ideas and thinking clearly on paper but, as with any skill, the more they practice the easier it will become.”[7]

Classroom Activities

Lesson 1: Interactive Math Notebook

Essential Question: What does multiplication mean to you? What is multiplication? What are the similarities and differences between addition and multiplication? Are multiplication and division related?

Background Information: Equal groups-unknown product, group size unknown, number of groups unknown. Arrays-unknown product, unknown factors. Area Model- unknown product, unknown factors. Comparison problems-unknown product, unknown factors.

Instruction: Using their math journals, students will create meaning of the different multiplication problem types. I will give them the equations and students will work to make sense of the equations.

Activity: Students will make journal entries as each new multiplication strategy is taught. Students will use numbers, words, and pictures to make sense of the topic. These entries will serve as math models and study aids to the students.

Assessment: The assessment will be a formative assessment. As students complete the journal activities, I will be able to use observation to assess their level of understanding. The journal can be used over and over again as a study or resource tool for the students.

Lesson 2: Create Your Own Problems

Essential Question: How do you use multiplication in your own life?

Background Information: Multiplication types: Equal groups-unknown product, group size unknown, number of groups unknown. Arrays-unknown product, unknown factors. Area Model- unknown product, unknown factors. Comparison problems-unknown product, unknown factors.

Instruction: Students will have completed an interactive math notebook which details the different multiplication types. They will then use this notebook as a resource to create their own problems.

Activity: Students will be given an index card with a multiplication equation written on it. The students will create a story problem for several different multiplication types. Students will choose one multiplication type themselves and they will be given a different type. After the students have created their problems, they will be given a partner and they will exchange problems. The goal is for their partner to be able to solve their story problem using the original multiplication equation.

Assessment: The assessment will be a formative assessment. As students complete the activity, I will be able to use observation to assess their level of understanding. As the partners solve each other's problems, they will be able to self-assess their work.

Lesson 3: Gallery Walks

Essential Question: Why do different strategies still come up with the same answer?

Background Information: Multiplication types: Equal groups-unknown product, group size unknown, number of groups unknown. Arrays-unknown product, unknown factors. Area Model- unknown product, unknown factors. Comparison problems-unknown product, unknown factors.

Instruction: Students will work in math circles, each student will be given a role. They will be working on developing problem solving techniques. Students will need to make sense of the problems and persevere while solving them. As they work in their groups, student will also create arguments and critique the reasoning of others.

Activity: Students will form small, math circle groups. Each student will choose a role and will be responsible for engaging with the problem. There will be large paper placed throughout the room. Each group will begin in front of one word problem involving multiplication. They will be given 10 minutes to work on the problem. There group will be responsible for making sense of the problem and then choosing a strategy to solve the problem. When the signal sounds, they will put their solution on the large poster paper and leave their paperwork with the problem and move to the next problem. Here they will look over the strategy and solution the group before them has used. The will critique the work of their peers and then decide on a different strategy to solve the same problem. They will reflect on their work and participate in a discussion about the solutions. Were the solutions the same? Why or why not? Students will repeat his process a third time. We will then come together as a class to discuss what we have discovered during this activity.

Assessment: The assessment will be a formative assessment. As students complete the activity, I will be able to use observation to assess their level of understanding. As the groups solve each other's problems, they will be able to self-assess their work.

Appendix A

The following is a list of vocabulary words that will benefit students during multiplication instruction. This is not meant to be a complete list, everyone should add or delete any terms they feel will be beneficial in their classrooms.

Vocabulary

multiplication

multiplicand

multiplier

product

digit

equal groups

partitioning

fair shares

factor

equation

repeated Addition

area model

array

column

row

proof

reasonable

Commutative Property

Associative Property

Distributive Property

Appendix B

Common Core State Standards- Mathematics

Represent and solve problems involving multiplication and division.

CCSS.MATH.CONTENT.3.OA.A.1
Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as 5 × 7.*

CCSS.MATH.CONTENT.3.OA.A.2
Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.*

CCSS.MATH.CONTENT.3.OA.A.3
Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]

CCSS.MATH.CONTENT.3.OA.A.4
Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = _ ÷ 3, 6 × 6 = ?*

Understand properties of multiplication and the relationship between multiplication and division.

CCSS.MATH.CONTENT.3.OA.B.5
Apply properties of operations as strategies to multiply and divide.[2] *Examples: If 6 × 4 =*

24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.)

CCSS.MATH.CONTENT.3.OA.B.6
Understand division as an unknown-factor problem. *For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.*

Multiply and divide within 100.

CCSS.MATH.CONTENT.3.OA.C.7
Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

Solve problems involving the four operations, and identify and explain patterns in arithmetic.

CCSS.MATH.CONTENT.3.OA.D.8
Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[3]

CCSS.MATH.CONTENT.3.OA.D.9
Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends*.

CCSS.MATH.CONTENT.3.NBT.A.3
Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

Mathematical Practices

CCSS.MATH.PRACTICE.MP1 Make sense of problems and persevere in solving them.

CCSS.MATH.PRACTICE.MP2 Reason abstractly and quantitatively.

CCSS.MATH.PRACTICE.MP3 Construct viable arguments and critique the reasoning of others.

CCSS.MATH.PRACTICE.MP4 Model with mathematics.

CCSS.MATH.PRACTICE.MP5 Use appropriate tools strategically.

CCSS.MATH.PRACTICE.MP6 Attend to precision.

CCSS.MATH.PRACTICE.MP7 Look for and make use of structure.

CCSS.MATH.PRACTICE.MP8 Look for and express regularity in repeated reasoning.

Appendix C

Discussion Questions Starters to help support the Mathematical Practices

Make Sense of problems and persevere in solving them

What is the problem asking?

How will you use that information?

What other information do you need?

Why did you choose this operation?

What is another way to solve this problem?

What did you do first? Why? How did you decide to do that first?

What can you do if you don’t know how to solve a problem?

Have you solved a problem similar to this one? How can it help?

When did you realize your first method would not work for this problem?

How do you know your answer makes sense?

Reason abstractly and quantitatively

What is a situation that could be represented by this equation?

Why does this operation represent this situation?

What is another operation you could have used to represent the situation?

What properties did you use to find your answer?

How do you know your answer is reasonable?

How can we show that this is true for all cases?

In what cases might our conclusion not hold true?

How can we verify this answer?

Construct viable arguments and critique the reasoning of others

Will that method always work?

How do you know?

What do you think about what he/she said?

Who can tell us about a different method?

What do you think will happen if…?

When would that not be true?

Why do you agree/disagree with what he/she said?

What do you want to ask her/him about that method?

How does that drawing support your work?

Explain the reasoning behind your prediction.

Show how you know that this statement is true.

How could we check that solution?

Model with mathematics

Why is that a good model for this problem?

How can you use a simpler problem to help you find the answer?

What conclusions can you make from your model?

How would you change your model if…?

Use appropriate tools strategically

What could you use to help you solve the problem?

What strategy could you use to make that calculation easier?

How would you estimate to help you solve that problem?

Why did you decide to use… to solve this problem?

Attend to precision

How do you know your answer is reasonable?

How can you use math vocabulary in your explanation?

How do you know those answers are equivalent?

What does that mean?

Look for and make use of structure

How did you discover that pattern?

What other patterns can you find?

What rule did you use to make this group?

Why can you use that property in this problem?

How is that like…?

Look for and express regularity in repeated reasoning

What do you remember about…?

What happens when…?

What if you…instead of…?

What might be a shortcut for…?

APPENDIX D RESOURCES

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[1] (Dobson n.d.)
[2] (College and Science 2015)
[3] (Ballinger n.d.)
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[7] (LLC 2015)

Curriculum Unit Title: Moving Third Graders From Guessing and Checking to Reasoning and Proof for Understanding

Author: Kathleen G. Gormley

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Students will be able to connect addition strategies to multiplication strategies. Students will be able to use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. Students will be able to Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. Students will be able to apply properties of operations as strategies to multiply and divide.

ESSENTIAL QUESTION(S) for the UNIT

What is multiplication? What are the similarities and differences between addition and multiplication? Are multiplication and division related? What does multiplication mean to you? How do you use multiplication in your own life? Why do different strategies still come up with the same answer?

CONCEPT A	CONCEPT B	CONCEPT C
Creating meaning for multiplication by creating an interactive journal	Student create their own multiplication word problems.	Students compare strategies to Construct viable arguments and critique the reasoning of others.
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
What is multiplication? What are the similarities and differences between addition and multiplication? What does multiplication mean to you?	How do you use multiplication in your own life?	Why do different strategies still come up with the same answer? Which strategy works best? How do you know?
VOCABULARY A	**VOCABULARY B**	**VOCABULARY C**
Product, factors, multiplicand, multiplier, equal groups, arrays, area model,	Multiplication, multiplicand, multiplier, product, digit, equal groups, partitioning, fair shares, factor, equation, repeated addition, area model, array, column , row, proof, reasonable, Commutative Property, Associative Property, Distributive Property	Multiplication, multiplicand, multiplier, product, digit, equal groups, partitioning, fair shares, factor, equation, repeated addition, area model, array, column , row, proof, reasonable, Commutative Property, Associative Property, Distributive

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

Concept A: Background Information: Equal groups-unknown product, group size unknown, number of groups unknown. Arrays-unknown product, unknown factors. Area Model- unknown product, unknown factors. Comparison problems-unknown product, unknown factors. Instruction: Using their math journals, students will create meaning of the different multiplication problem types. I will give them the equations and students will work to make sense of the equations. Activity: Students will make journal entries as each new multiplication strategy is taught. Students will use numbers, words, and pictures to make sense of the topic. These entries will serve as math models and study aids to the students. **Concept B:** Background Information: Multiplication types: Equal groups-unknown product, group size unknown, number of groups unknown. Arrays-unknown product, unknown factors. Area Model- unknown product, unknown factors. Comparison problems-unknown product, unknown factors. Instruction: Students will have completed an interactive math notebook which details the different multiplication types. They will then use this notebook as a resource to create their own problems. Activity: Students will be given an index card with a multiplication equation written on it. The students will create a story problem for several different multiplication types. Students will choose one multiplication type themselves and they will be given a different type. After the students have created their problems, they will be given a partner and they will exchange problems. The goal is for their partner to be able to solve their story problem using the original multiplication equation. **Concept C:** Students will work in math circles, each student will be given a role. They will be working on developing problem solving techniques. Students will need to make sense of the problems and persevere while solving them. As they work in their groups, student will also create arguments and critique the reasoning of others.