Meaningful Operations with Rational Expressions through Analogical Reasoning

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Objectives

I have been teaching high school math for more than 20 years and too many students, even at the highest levels, tell me "I can't do fractions." It certainly doesn't get any better when we talk about sets of numbers and refer to fractions as rational numbers. In my readings, I learned that I am not alone. Middle and high school teachers and college professors have observed weaknesses in students' ability to operate on rational numbers for decades. That makes teaching algebra units on rational expressions and equations a real challenge. This curriculum unit addresses three major sources of confusion - vocabulary and terminology, operations with rational numbers, and the connection between rational numbers and rational expressions. While I don't expect miracles, at the end of the unit, students should have improved number sense from a greater comfort level with rational numbers, as well as more confidence with algebraic manipulations.

I teach in a vocational school district consisting of four high schools that draw from middle schools in all five districts in New Castle County, Delaware so that students arrive having used a variety of curricular programs. My school is a "choice" public school and our students are held to the same academic standards as all public school students in the state. We use the *Core-Plus: Contemporary Mathematics in Context* (CPMP) textbook series, an integrated math curriculum. We teach math courses in 90-minute block periods every day for one 18-week semester; students have three to four courses per semester, including their chosen career area for 90 – 180 minutes per day for the entire school year. With some exceptions for the very high and very low achieving freshmen, based on a district placement test, students are grouped heterogeneously and we can have a wide range of abilities in each class. Students currently take two semesters of *Core-Plus* (1 & 2) in their freshman year, *Core-Plus* 3 in their sophomore year, and a "Trigonometry" transition course in their junior year. The "Trigonometry" course (although it includes other topics, as well), may serve as a capstone course, or to prepare underclassmen for Pre-calculus. Despite four prior high school math courses, I still see weaknesses in number sense and algebraic manipulations when students reach Pre-calculus.

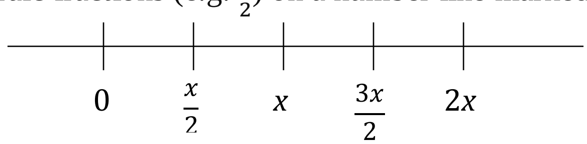
I am writing this curriculum unit for my Pre-calculus students, but it would be appropriate for Algebra I or II students, as well. This unit will address the Common Core State Standards (CCSS) related to rational expressions. Specifically, HSA.APR.D.7 requires students to "understand that rational expressions form a system ***analogous*** to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions." The keyword in the standard that connects this curriculum unit to my seminar "Mathematical Proof and Reasoning" is the word *analogous*. In this unit, students will very deliberately compare operations with rational expressions to analogous operations with rational numbers (AKA fractions) to evaluate the validity of each step. In addition to the content being studied, this unit will also emphasize two of Math Practice Standards: "Construct viable arguments and critique the reasoning of others" and "Reason abstractly and quantitatively." In the end, I hope students feel better prepared for the 11th grade Smarter Balance test and the new SAT.

Rationale

Articles that I read in preparation for this unit confirmed some of my thoughts about the causes of students' errors with rational numbers and expressions. I'm sure that every algebra teacher has seen students cross out numerals and/or variables that they see in both the numerator and denominator of a fraction whether they are common factors of each, or not. One source of errors is based on inappropriate operations on factors and terms. I am convinced that my students don't really understand the definition of factors, and/or how it applies to rational numbers (or polynomials, either). Students sometimes confuse processes they learned for addition and multiplication; for example, they may find a common denominator before multiplying. Sometimes they apply a factor to both the numerator and denominator $\left(i.e. -2 \cdot \frac{4}{5} = \frac{-8}{-10}\right)$, or make errors when finding equivalent fractions.

In other articles I read, authors place some blame on inaccurate terminology we teachers use. In 1924, August Grossman wrote, "pupils are apt to cancel any two numbers or expressions that are the same, no matter in what context."[1] In 1988, William Blubaugh wrote about "the vagueness and misconceptions that are associated with the word cancel (which to many students means no more than to scratch out)."[2] In 1999, Ted Szymankski stated that the process of *canceling* "gets more students into trouble than any other activity in basic mathematics."[3] Errors in crossing out numbers that are the same without considering why is obviously not something new to todays students, yet I am still going to attempt to correct their misconceptions in this unit.

Some middle school teachers attribute the difficulty that students have when moving from arithmetic to algebra to difficulties with fractions. Joy Darley writes, "Since our goal is for students to better understand variables, we need to be certain that they understand numbers."[4] She demonstrates the placement of fractions on a number line in parallel to algebraic fractions (e.g. $\frac{x}{2}$) on a number line marked in multiples of *x*:



Darley and Leapard later wrote, "…one could possess knowledge of fraction properties and a related topic in algebra but be unable to relate the two. Once the connection is made between the two, students will be more confident using variables."[5] I see this last statement as the rationale for using the Mathematical Reasoning and Proof techniques that I learned in my seminar and research to help my students understand operations involving rational expressions. I envision students searching for counterexamples to identify or confirm incorrect/invalid operations (as in critiquing the reasoning of others), and using Analogical Reasoning based on the Properties of Arithmetic to justify operations and procedures.

Analogical Reasoning

I was surprised and encouraged to learn that there is a name for the type of mathematical reasoning I had in mind for my students in this unit. It's called Analogical Reasoning and is based on students making analogies between previously learned concepts and new ones. In this unit, I will explicitly help my students make analogies between rational numbers and rational expressions. An *analogy* is a comparison between two things that highlights their similarities, and "*analogical reasoning* is any type of thinking that relies upon an analogy. An *analogical argument* is an explicit representation of a form of analogical reasoning that cites accepted similarities between two systems to support the conclusion that some further similarity exists."[6]

Researchers describe the use of analogies in making sense of observations and making inferences from them. Dedre Gentner describes how analogies are used to explain new concepts first by retrieving information from long-term memory and second by mapping it according to its similarities and structure. In her article, Gentner states "Connected systems are easier to map to a new domain than are unconnected sets, leading to better transfer in analogy and problem-solving."[7] This is exactly what I had in mind when I began thinking about this curriculum unit. Gentner has conducted much more research about what is happening in the brain during the mapping process for multiple content areas that can be found in her article.

According to Paul Bertha, Aristotle wrote about two forms of arguments: argument from example and argument from likeness. They may be the predecessors of analogical arguments. Another philosopher named David Hume also wrote about using similarities and differences between objects to reason analogically and make inferences. Also in his article, Bartha outlines the form of an analogical argument. For the source domain *S* and target domain *T*,

1. *S* is similar to *T* in certain (known) respects.
2. *S* has some further feature *Q*.
3. Therefore, *T* also has the feature *Q*, or some feature *Q** similar to *Q*.[8]

Later, Bartha uses a table to represent the source attributes in one column next to the target attributes in another, in a *horizontal relationship*, to see similarities or differences between the two. The similar attributes of the source and the target lead to, *vertically*, a plausible inference/hypothesis that the target has an *inferred similarity*. It's probably important to note that analogical reasoning is a form of inductive reasoning, "since their conclusions do not follow with certainty but are only supported with varying degrees of strength."[9]

In the activities in this curriculum unit students will use the table format to show side-by-side comparisons (horizontal relationship) of operations on rational expressions with variables in one column and with numbers substituted for the variables in the second column. Example 3, below, illustrates four basic arithmetic operations for rational expressions and rational numbers. Students will produce similar tables for each step in the process of simplifying, multiplying and dividing rational expressions (requiring factoring and dividing by common factors), and adding and subtracting rational expressions (requiring finding common denominators and equivalent fractions). In this way, students will justify their solutions, including the steps along the way.

Background Content

Definitions of Rational Numbers and Rational Expressions

Rational Numbers, represented by the symbol $\mathbb{Q}$, are a set of numbers that can be written as the ratio of two integers: $\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \text{ are integers}, q \neq 0\right\}$. To students, we are renaming the set of numbers they know as *fractions*. They may not like them or feel comfortable working with them, but they are familiar with them. They should understand that the set of rational numbers is *closed* under addition, subtraction, multiplication and division (as long as we don't divide by zero). A set is *closed* with respect to an operation if the inputs and the outputs are all members of the set. Example 1 demonstrates that the sum, difference, product and quotient of two rational numbers (inputs) are also rational numbers (outputs):

Example 1: $\frac{4}{3} + \frac{2}{3} = \frac{6}{3}$ $\qquad \frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9}$

$\frac{4}{3} - \frac{2}{3} = \frac{2}{3}$ $\qquad \frac{4}{3} \div \frac{2}{3} = \frac{2}{1}$

In Algebra, we introduce variables and perform arithmetic operations on polynomials, in the form $p(x) = a_1x^n + a_2x^{n-1} + \cdots + a_nx + a_{n+1}$, where *a* and *x* are any real number, and *n* is an integer representing the degree of the polynomial. Example 2 shows that the set of polynomials is also *closed* under addition, subtraction and multiplication;

the inputs and outputs are all polynomials.

Example 2: $(4x^3 - 3x^2 - 2x + 1) + (6x^2 - 5) = 4x^3 + 3x^2 - 2x - 4$
$(4x^3 - 3x^2 - 2x + 1) - (6x^2 - 5) = 4x^3 - 9x^2 - 2x + 6$
$(4x^3 - 3x^2 - 2x + 1)(6x^2 - 5) =$
$24x^5 - 18x^4 - 32x^3 + 21x^2 + 10x - 5$

The quotient of two polynomials, $\frac{a(x)}{b(x)}$, for $b(x) \neq 0$ is known as a rational expression, analogous to a rational number. (Some texts and articles also refer to them as algebraic fractions.) Like rational numbers, rational expressions are *closed* under addition, subtraction, multiplication and division of non-zero polynomials. Example 3 illustrates the four operations on rational expressions producing new rational expressions, as well as analogous operations on rational numbers by substituting the value of 3 for the variable, x. As stated above, this example also illustrates some of what I will expect my students to do to justify the validity of their steps in operations with rational expressions.

Example 3:

RATIONAL EXPRESSION OPERATIONS (x ≠ -1, -2)	RATIONAL NUMBER OPERATIONS (x = 3)
$\frac{4}{x+2} + \frac{2x}{x+1} = \frac{2x^2+8x+4}{(x+2)(x+1)}$	$\frac{4}{3+2} + \frac{2(3)}{3+1} = \frac{2(3)^2+8(3)+4}{(3+2)(3+1)} = \frac{46}{20}$
$\frac{4}{x+2} - \frac{2x}{x+1} = \frac{-2x^2+4}{(x+2)(x+1)}$	$\frac{4}{3+2} - \frac{2(3)}{3+1} = \frac{-2(3)^2+4}{(3+2)(3+1)} = -\frac{18}{20}$
$\frac{4}{x+2} \cdot \frac{2x}{x+1} = \frac{8x}{(x+2)(x+1)}$	$\frac{4}{3+2} \cdot \frac{2(3)}{3+1} = \frac{8(3)}{(3+2)(3+1)} = \frac{24}{20}$
$\frac{4}{x+2} \div \frac{2x}{x+1} = \frac{4(x+1)}{2x(x+2)} = \frac{2(x+1)}{x(x+2)}$; (x≠0, -1, -2)	$\frac{4}{3+2} \div \frac{2(3)}{3+1} = \frac{4}{5} \cdot \frac{4}{6} = \frac{2(3+1)}{3(3+2)} = \frac{8}{15}$

Sources of Errors

Factors versus Terms

Way back in 1988, Joseph Martinez wrote an article in *The Mathematics Teacher* entitled "Helping Students Understand Factors and Terms."[10] Martinez stated a need to provide instructional activities to help students understand the difference between factors and terms, and also how to write numbers and expressions in equivalent forms using Properties of Arithmetic and the proper Order of Operations. Martinez constructed a series exercise sets that addresses common mistakes students make when working with fractions, one concept at a time. The exercise sets guide students to both explore and articulate their ideas and conclusions about each one. The exercises begin with simple arithmetic and gradually move to algebraic fractions, reinforcing concepts as students progress through the problem sets. He employs horizontal relationships to compare and contrast; some exercises place addition and multiplication side-by-side to highlight their differences and others place arithmetic next to similar algebraic ones to highlight analogous characteristics. As students are asked to verbalize their thought processes and conclusions they are executing CCSS Mathematical Practices #2 and #3. As Martinez states, "When students are shown the relationship between an exercise with simple values and an exercise with unknowns, they are learning the principles behind the exercise rather than learning to work individual exercises."[11]

The succession of activities in this curriculum unit will, in part, be developed based on the guidelines proposed by Martinez. I think my students' understanding of *factors* versus *terms* will also improve after answering questions similar to some posed by August Grossman in 1924:

1. How many factors does the expression $4x(n + p)$ have, and what are they?
2. How many terms does the expression $4x(n + p) + 9 – 3abc$ have, and what are they?
3. What does 4 multiply in $4x(n + p) + 9 – 3abc$?

The expression in the first question has three factors being multiplied: 4, x, and $(n + p)$. It is important for students to see $(n + \text{p})$ as a single number being multiplied. In the second question, there are three terms: $4x(n + p)$ is one term, 9 is a second term and $3abc$ is a third term, each term being separated by a plus or minus sign. A third type of question emphasizes the fact that factors apply *only* within individual terms. The number 4 multiplies two other factors in the first term, $x(n + p)$; the 4 is *not* distributed to 9 or $3abc$. These questions focus on the definitions of *factors* and *terms*, in order to prepare students to recognize them in rational expressions.

Properties of Arithmetic and Order of Operations

There are properties of arithmetic that apply to the set of rational numbers, and to polynomials and rational expressions. Most high school students are aware of these properties, and use them, even if they do not know them by name. The Properties of Addition and Multiplication are 1) *Associative Property*: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all numbers a, b, c, 2) *Commutative Property*: $a + b = b + a$ and $a \cdot b = b \cdot a$ for all numbers a, b, c , 3) *Identity Property*: $a + 0 = a$ and $a \cdot 1 = a$ for all numbers a, 4) *Inverse Property*: $a + (-a) = 0$ and $a \cdot \left(\frac{1}{a}\right) = a$ for all non-zero numbers a, and 5) *Distributive Property* of multiplication over addition: $a(b + c) = ab + ac$ for all numbers a, b, c. More and more, I think we do our students a disservice by not reminding them explicitly about the use of these properties more frequently. If these properties were forced to the forefront of their brains more frequently, they may make fewer of the errors shown in Example 4 below.

All students were taught the proper order of operations – perform operations inside Parentheses first, followed by Exponents, Multiplication and Division, and finally Addition and Subtraction. Some students learned the pneumonic PEMDAS, and others learned "Please Excuse My Dear Aunt Sally." However, many think multiplication *always* goes before division and are surprised when I show them that division can be done first, and is often preferred, especially if it simplifies the arithmetic. The same is true with addition and subtraction; an expression can be rearranged using the commutative and associate properties to put numbers together to, again, simplify the arithmetic. I see my students make too many errors with respect to the order of operations. These errors will lead to errors in factoring and simplifying rational expressions. Some common errors that I hope to eliminate with the use of Analogical Reasoning are shown in Example 4.

Example 4:

ALGEBRAIC EXPRESSION	NUMERICAL EXPRESSION (x = 3)
$4(x + 2) \neq 4x + 2$	$4(3 + 2) = 20 \neq 4(3) + 2 = 14$
$-2x + 1 \neq 2x - 1$	$-2(3) + 1 = -5 \neq 2(3) - 1 = 5$
$x \cdot \frac{1}{x} = \frac{x}{x} \neq 0$	$3 \cdot \frac{1}{3} = \frac{3}{3} = 1 \neq 0$
$5 \cdot \frac{2}{x} \neq \frac{5}{5} \cdot \frac{2}{x} = \frac{10}{5x}$	$5 \cdot \frac{2}{3} = \frac{10}{3} \neq \frac{5}{5} \cdot \frac{2}{3} = \frac{10}{15} = \frac{2}{3}$
$(x + 5)^2 \neq x^2 + 5^2$	$(3 + 5)^2 = 64 \neq 3^2 + 5^2 = 34$
$(-x)^2 \neq -x^2$	$(-3)^2 = 9 \neq -3^2 = -9$
$2x^3 + 5x^2 \neq (2x)^3 + (5x)^2$	$2 \cdot 3^3 + 5 \cdot 3^2 = 99$ $\neq (2 \cdot 3)^3 - (5 \cdot 3)^2 = 441$

Cancelling

We teachers often use the term *canceling* in multiple contexts. We use it to subtract the same number from each side of an equation. We use it to cross out common factors in fractions. However, canceling is not even an operation; instead of using the word "cancel," we should use accurate terminology to describe the operation we are performing. In the case of solving equations, we should explicitly tell (and show) students that we are adding or subtracting the same value from each side of an equation to keep it balanced by use of the Additive Identity, 0. In the case of simplifying rational numbers or rational expressions, we should explicitly tell students we are *dividing* by a common factor, demonstrating that a common factor in both the numerator and denominator is equal to the Multiplicative Identity, one. I was somewhat amused by, yet agree with, Arron Eisen from Toronto[12] with regard to the acronym he named FFOO – Fancy Form Of One. By constantly writing the number one in multiple forms, we can repeatedly emphasize how we are multiplying by the Identity when we simplify fractions, find equivalent fractions with a common denominator, and rationalize denominators, etc.

In his article, August Grossman recommends four steps to guide students to understanding valid operations with algebraic fractions. Still working with the term *cancellation*, the author first uses examples to define the operation as the "reverse of the one effected by the quantity cancelled." For example, given *2x*, cancelling 2 would mean division, and given, *x+5*, cancelling 5 would mean subtraction. Grossman's second step is to remind pupils that the value of a fraction cannot change when cancellation takes place. He demonstrates the difference between multiplication or division in both the numerator and denominator versus addition or subtraction in both the numerator and denominator with numerical examples: $\frac{5(2)}{8(2)} = \frac{5}{8}$ and $\frac{12\div3}{15\div3} = \frac{12}{15}$, but $\frac{5+2}{8+2} \neq \frac{5}{8}$ and $\frac{9-1}{10-1} \neq \frac{9}{10}$. The purpose of the examples is to emphasize that multiplying or dividing by the same number in both the numerator and denominator does produce equivalent fractions while addition and subtraction do not. Grossman's third step, titled "How far does the effect of a multiplier extend?" addresses the concept of factor versus term. He summarizes his examples with "the force of any factor extends only to the + (or -) sign." In other words, factors operate within terms only, and terms in an expression are separated by the + (or -) sign. I view this step as blending terminology with the order of operations which would solidify student understanding of *factor versus term* even further. In his final step, Grossman answers the question, "When can you cancel?" by putting together the first three steps into more complex fractions.

I intend to incorporate strategies to correct student misconceptions about canceling from Grossman, Blubaugh and Szymanski's articles into the activities for this unit. Since reading the articles, I personally, have tried to stop using the word *cancel* in my classes. I told my students why, and find myself having to stop and rephrase what I am saying to use proper terminology. It's a challenge, but I think my students are beginning to see that using the proper words helps avoid confusion.

Miscellaneous Common Errors

When working with fractions or rational expressions, there are still many more sources of errors. I am acknowledging them here, but they will not be the main focus of the activities. As evaluated by W. Storer and the Board of Studies for Mathematics, some errors may be algebraic while others extend to arithmetic – all the mistakes students make with fractions. I do hope that students will find and correct some of their own mistakes by justifying their steps when working through problems using analogical reasoning in tabular form.

In a typical algebra unit, students begin by factoring the numerators and denominators of rational expressions and looking for common factors that can be divided. Errors may arise if students do not factor correctly, or completely. Students need to recognize when a rational expression is equal to -1, as in $\frac{4-n}{n-4} = \frac{-1(n-4)}{n-4} = -1$. Otherwise, they will not simplify an expression completely.

After factoring and simplifying single rational expressions, students practice multiplication and division of two expressions. Errors can occur if students don't rewrite division as *multiplication by the reciprocal* properly, or if they factor incorrectly, or if they divide common factors between two expressions before inverting the second one.

By far, the operations that produce the most errors are addition and subtraction. They require factoring, finding a common denominator, converting to equivalent expressions with the common denominator and, finally, adding or subtracting the numerators (combining like terms, assuming all prior steps were done correctly). Extra care must be taken with subtraction to subtract *all* terms in the numerator (applying the distributive property using -1). In some cases, it is also necessary to factor the final numerator and look for common factors in the denominator. Some simple/careless errors can be caught by substituting values for the variables in the second column of an analogical reasoning table, as shown in Example 5.

Example 5:

RATIONAL EXPRESSION	RATIONAL NUMBER (x=3)
$\frac{2}{x^2-25} - \frac{1}{x^2+5x}$	$\frac{2}{3^2-25} - \frac{1}{3^2+5\cdot 3} = \frac{2}{-16} - \frac{1}{24}$
$\frac{2}{(x-5)(x+5)} - \frac{1}{x(x+5)}$	$\frac{2}{(3-5)(3+5)} - \frac{1}{3(3+5)}$ $= \frac{2}{-16} - \frac{1}{24}$
$\frac{2}{(x-5)(x+5)} \cdot \frac{x}{x} - \frac{1}{x(x+5)} \cdot \frac{(x-5)}{(x-5)}$	$\frac{2}{(3-5)(3+5)} \cdot \frac{3}{3} - \frac{1}{3(3+5)} \cdot \frac{(3-5)}{(3-5)} = \frac{6}{-48} - \frac{-2}{-48}$
$\frac{2x}{x(x-5)(x+5)} - \frac{(x-5)}{x(x-5)(x+5)}$	$\frac{2\cdot 3}{3(3-5)(3+5)} - \frac{(3-5)}{3(3-5)(3+5)} = \frac{6}{-48} - \frac{-2}{-48}$
$\frac{2x-x+5}{x(x-5)(x+5)}$	$\frac{2\cdot 3-3+5}{3(3-5)(3+5)} = \frac{8}{-48}$

$\frac{x+5}{x(x-5)(x+5)} = \frac{1}{x(x-5)}$	$\frac{8}{-48} = \frac{1}{3(3-5)} = \frac{1}{-6}$

Solving Rational Equations

Students initially learn to operate with rational expressions in algebra classes, or in units in an integrated program. They generally see exercises that allow them to practice skills like factoring and combining polynomials. Later, students are asked to solve rational equations and graph rational functions. In both of these cases, we have to pay more attention to the domain, restricting it to values that ensure nonzero denominators. After solving an equation, we must check that the solution is part of the domain and not an extraneous solution. When graphing, the domain helps identify vertical asymptotes.

I read an interesting article by Larry Orzechowski about the process of solving rational equations. The most common method taught involves multiplying all terms in the equation by the lowest common denominator to eliminate fractions and then solve the remaining equation. The point Orzechowski makes is that this method is completely different from all of the operations students practiced with rational expressions. He suggests solving the equation by rewriting the equation with equivalent terms having a common denominator. After adding or subtracting to form a single rational expression on each side of the equation, it is clear that the numerators must be equal if they have the same denominator. At this point, students end up solving the same equation as the one obtained by eliminating denominators,[13] but they will be more likely to understand how to get to it. I pose an example of this procedure in Example 6. I think it may benefit my students to see the original operations put to use in a different context.

Example 6: Solve $\frac{x+4}{x-2}+\frac{6}{x-2}=\frac{1}{x+3};\ x \neq 2,-3$

$$\frac{x+4}{x-2}\cdot\frac{x+3}{x+3}+\frac{6}{x-2}\cdot\frac{x+3}{x+3}=\frac{1}{x+3}\cdot\frac{x-2}{x-2}$$

$$\frac{x^2+7x+12}{(x-2)(x+3)}+\frac{6x+18}{(x-2)(x+3)}=\frac{x-2}{(x-2)(x+3)}$$

$$\frac{x^2+13x+30}{(x-2)(x+3)}=\frac{x-2}{(x-2)(x+3)}$$

Since the denominators are the same, the numerators must be equal.

$x^2 + 13x + 30 = x – 2$
$x^2 + 12x + 32 = 0$
$(x + 4)(x + 8) = 0$
$\{-4, -8\}$

Both solutions are part of the domain.

I had the opportunity to try Orzechowski's idea before completing the writing of this curriculum unit. After several days of simplifying, multiplying, dividing, adding, and subtracting rational expression, I simply presented a rational equation on my SMART Board to two different classes and asked students to solve it. Every student got to work immediately and solved the equation without direct instruction. As I observed students at work, every one of them began by finding a common denominator for all terms on both sides of the equation. Some students recognized right away that the only way two rational expressions (on two sides of an equation) with the same denominator will be equal is if the numerators are equal. Once that fact was spread around the classroom, students were able to find the solution(s). The only instruction my students required was demonstrating the need to test all solutions to be sure they are part of the domain of the original expression and not extraneous solutions. I am thoroughly convinced that rewriting all terms with a common denominator is a more natural way for students to solve rational equations.

Activities

Activity #1 – Factors versus Terms

Essential Question: How do we interpret parts of an expression, such as terms, factors, and coefficients? (CCSS.Math.Content.HSA.SSE.A.1.a)

This activity includes 3 sets of sample questions that I will use for extended warm-up problem sets the week before students begin studying rational expressions. I will repeat the same types of problems several consecutive days until I feel as though students have mastered the vocabulary. These problem sets could also be used as homework assignments. Additionally, since my school district has adopted *Schoology* as a Learning Management System, especially for Blended Learning cohorts using technology, I can assign additional practice for individual students that need it. Problem Set #1 focuses on identifying terms in simple and complex expressions. It also asks students to write their own expressions that satisfy certain criteria. Problem Set #2 focuses on identifying factors, including numerals, variables, and polynomials. Problem Set #3 will assess students' ability to interpret how factors apply to terms in an expression. It requires students to apply the Distributive Property and the Order of Operations appropriately. If students struggle with this problem set, I will introduce Analogical Reasoning and require them to substitute numbers for variables, side-by-side for each step, to test whether their expressions remain equivalent.

Problem Set #1 (with answers to the right)

How many *terms* are in each expression? *Name them.*

1. $6x^2 – 5x + 4$ (3 terms; $6x^2$, $– 5x$, 4)
2. $6(x^2 – 5x + 4)$ (1 term; $6(x^2 – 5x + 4)$)
3. $-4(x – 1)^2 + 13x – 2$ (3 terms; $-4(x – 1)^2$, $13x$, -2)
4. $10abd(3e – f)(4k + 17)$ (1 term; $10abd(3e – f)(4k + 17)$)
5. $\frac{15x-5}{10}$ (1 term; $\frac{15x-5}{10}$)
6. Write an expression with 2 terms.
7. Write an expression with 4 terms using 6 different variables.
8. Write an expression with 5 terms using 3 different variables.

Problem Set #2 (with answers to the right)

How many *factors* are in each expression? *Name them.*

1. $3 \cdot 8 \cdot 10npkh$ (7 factors; 3, 8, 10, n, p, k, h)
2. $10abd(3e – f)(4k + 17)$ (6 factors; 10, *a*, *b*, *d*, $(3e - f)(4k + 17)$)
3. $-60buw(n^3 – z^2)$ (5 factors; -60, *b, u, w*, $(n^3 – z^2)$)
4. $(b – c + 1)(d + y)(f – 2)$ (3 factors; $(b – c + 1)$, $(d + y)$, $(f – 2)$)
5. $-3(5 + x^2)(2y – cd)$ (3 factors; -3, $(5 + x^2)$, $(2y – cd)$)

Problem Set #3 (with answers)

1. What is the "6" multiplying in $6(x^2 – 5x + 4)$? Rewrite the expression as only the sum of terms.
   (The 6 is multiplying the single factor in parentheses. Applying the Distributive Property, the expression can be rewritten as $6x^2 – 30x + 24$.)
2. What is "-3" multiplying in $-3(5 + x^2)(2y – cd)$? Rewrite the expression as only the

sum of terms.
(The -3 is multiplying the <u>product</u> of the other two factors in parentheses; it is only multiplied <u>once</u>. It can be "distributed" to each term inside one set of parentheses, but <u>not</u> both. In expanded form, the expression is equal to $-30y - 6x^2y + 15cd + 3x^2cd$.)

3. What is the "-4" multiplying in $-4(x - 1)^2 + 13x - 2$? Rewrite the expression as only the sum of terms, in simplest form.
(The -4 is multiplying $(x - 1)^2 = x^2 - 2x + 1$ only. The equivalent expression in simplest form is $-4x^2 + 21x - 6$.)

The first column in the table below demonstrates some common errors that I have observed students make with exponents, and are likely to make with Question #3 above. They often multiply a coefficient before squaring a variable, thereby squaring the coefficient also. Another common error occurs when squaring a binomial; students too often square just the individual terms and ignore the other two identical products of the two terms. In other words, $(a + b)^2 = a^2 + b^2 + ab + ab$, <u>not</u> $(a + b)^2 = a^2 + b^2$. The second column illustrates the use of Analogical Reasoning to help students see the expressions are not equivalent, and hopefully find their own errors.

ALGEBRAIC EXPRESSION ERRORS	NUMERICAL EXPRESSION x = 3
$-4(x - 1)^2 = (-4x + 4)^2$	$-4(3-1)^2 = -16 \neq (-4\cdot 3 + 4)^2 = (-8)^2 = 64$
$-4(x - 1)^2 = 16x^2 + 16$	$-4(3-1)^2 = -16 \neq 16\cdot 3^2 + 16 = 160$
$-4(x - 1)^2 = -4(x^2 - 1)$	$-4(3-1)^2 = -16 \neq -4(3^2 - 1) = -32$
$-4(x - 1)^2 = -4(x^2 + 1)$	$-4(3-1)^2 = -16 \neq -4(3^2 + 1) = -40$

Activity #2 – Simplifying Rational Numbers and Expressions

Essential Question: In what ways are rational expressions analogous to rational numbers? (CCSS.Math.Content.HSA.APR.D.7)

This activity includes 6 sets of sample questions that are intended to highlight valid operations for simplifying fractions and rational expressions. If students have already been exposed to rational expressions, these problem sets can be used as warm-ups or homework. However, if it is a new topic of study I will use them sequentially prior to moving to traditional textbook problems for simplifying rational expressions. The purpose of Problem Set #4 is to remind students of the correct methods for adding and multiplying fractions. Problem Sets #5-7 are intended to emphasize what common factors *do* and *do no*t look like in rational numbers and expressions. Problem Set #8 mixes terms and factors, serves as a formative assessment to determine whether students are ready to move on, and introduces Analogical Reasoning as a method for checking answers. Problem Set #9 utilizes a table format with traditional rational expression problems that require factoring and analogous rational numbers to verify the simplified forms are equivalent to the original.

Problem Set #4

Part I. Rewrite each rational expression as a sum of 3 terms, then simplify.

Rational Expression	**Sum of 3 Terms**
$\frac{3+2+3}{3}$	$\frac{3}{3}+\frac{2}{3}+\frac{3}{3}=\frac{8}{3}$
$\frac{3+2+2}{3}$	$\frac{3}{3}+\frac{2}{3}+\frac{2}{3}=\frac{7}{3}$
$\frac{3+3+3}{3}$	$\frac{3}{3}+\frac{3}{3}+\frac{3}{3}=1$
$\frac{3+4+5}{4}$	$\frac{3}{4}+\frac{4}{4}+\frac{5}{4}=3$
$\frac{4+4+4}{4}$	$\frac{4}{4}+\frac{4}{4}+\frac{4}{4}=3$

Part II. Rewrite each rational expression as a product of 3 factors, then simplify.

Rational Expression	**Product of 3 Factors**
$\frac{3\cdot 2\cdot 3}{3}$	$\frac{3}{3}\cdot\frac{2}{1}\cdot\frac{3}{1}=6$
$\frac{3\cdot 2\cdot 2}{3}$	$\frac{3}{3}\cdot\frac{2}{1}\cdot\frac{2}{1}=4$
$\frac{3\cdot 3\cdot 3}{3}$	$\frac{3}{3}\cdot\frac{3}{1}\cdot\frac{3}{1}=9$
$\frac{3\cdot 4\cdot 5}{4}$	$\frac{3}{1}\cdot\frac{4}{4}\cdot\frac{5}{1}=15$
$\frac{4\cdot 4\cdot 4}{4}$	$\frac{4}{4}\cdot\frac{4}{1}\cdot\frac{4}{1}=16$

These problems may appear simple and straightforward, yet when I posed them to my current students, I had several of them rewrite the rational numbers involving products (Part II) as the product of 3 numbers with the same denominators. It presented the opportunity to re-emphasize the difference between terms and factors, and that common factors can be divided within single terms (e.g. $\frac{3}{3}$) because it is the multiplicative identity.

Problem Set #5 (with answers)

Simplify. *Explain similarities or differences between A, B, C and D in each row.*

A	**B**	**C**	**D**

$\frac{3+3}{3}=\frac{6}{3}=2$	$\frac{x+x}{x}=\frac{2x}{x}=2$	$\frac{3\cdot 3}{3}=3$	$\frac{x\cdot x}{x}=x$
$\frac{3+3+3}{3}=\frac{9}{3}=3$	$\frac{3x+3x+3x}{3x}=\frac{9x}{3x}=3$	$\frac{3\cdot 3\cdot 3}{3}=3^2$	$\frac{3x\cdot 3x\cdot 3x}{3x}=(3x)^2$
$\frac{3+2+2}{3}=\frac{7}{3}$	$\frac{3x+2x+2x}{3x}=\frac{7x}{3x}=\frac{7}{3}$	$\frac{3\cdot 2\cdot 2}{3}=2^2$	$\frac{(3x)(2x)(2x)}{3x}=(2x)^2$
$\frac{3+5}{3+5}=\frac{8}{8}=1$	$\frac{x+y}{x+y}=1$	$\frac{3\cdot 5}{3\cdot 5}=\frac{15}{15}=1$	$\frac{x\cdot y}{x\cdot y}=\frac{xy}{xy}=1$
$\frac{4}{4+4}=\frac{4}{8}=\frac{1}{2}$	$\frac{x}{x+x}=\frac{x}{2x}=\frac{1}{2}$	$\frac{5}{5\cdot 5}=\frac{5}{25}=\frac{1}{5}$	$\frac{x}{x\cdot x}=\frac{x}{x^2}=\frac{1}{x}$

Students should recognize that the sums in columns A and B are identical; the variables in column B behave the same as numbers in column A. Likewise, the multiplication and division in columns C and D behave the same with numbers and variables, respectively. However, common factors can be divided (equal to one) quickly and easily in columns C and D, but an extra step is needed before dividing common factors when there is addition (or subtraction) in the numerator or denominator of rational expressions as in columns A and B.

Problem Set #6 (with answers)

Simplify. *Explain similarities or differences between A and B pairs.*

A	B
$\frac{3(2+3)}{6}=\frac{15}{6}=\frac{5}{2}$	$\frac{3(2\cdot 3)}{6}=\frac{18}{6}=3$
$\frac{4(8+12)}{12}=\frac{80}{12}=\frac{20}{3}$	$\frac{4(8\cdot 12)}{12}=32$
$\frac{6+3(2+3)}{6}=\frac{21}{6}=\frac{7}{2}$	$\frac{6\cdot 3(2\cdot 3)}{6}=18$
$\frac{3(2+3)+4(2+5)}{3(2+3)}=\frac{43}{15}$	$\frac{3(2\cdot 3)\cdot 4(2\cdot 5)}{3(2\cdot 3)}=40$
$\frac{3(2x+3x)}{6x}=\frac{15x}{6x}=\frac{5}{2}$	$\frac{3(2x\cdot 3x)}{6x}=\frac{18x^2}{6x}=3x$

While the numbers are the same between columns A and B, the operations are completely different. Common factors can be divided easily with only multiplication in the numerator in column B, but more work is required to find common factors when there are multiple terms in the numerator, as in column A. Analyzing student work on this problem set serves as a formative assessment for whether additional practice is required.

Problem Set #7 (with answers)

Simplify if possible. *Explain similarities or differences between A and B pairs.*

A	B
$\frac{5+2x}{2x}$; simplest form	$\frac{5\cdot 2x}{2x}=5$
$\frac{5+2x}{5}$; simplest form	$\frac{5\cdot 2x}{5}=2x$
$\frac{x+5}{5x}$; simplest form	$\frac{x\cdot 5}{5x}=1$
$\frac{5x+x}{5}=\frac{6x}{5}$	$\frac{5x\cdot x}{5}=x^2$
$\frac{5x+x}{5x}=\frac{6x}{5x}=\frac{6}{5}$	$\frac{5x\cdot x}{5x}=x$

Interestingly, the first three expressions in column A baffled some of my students because they thought they had to rewrite them. On the positive side, they knew there were no common factors unless they rewrote the expression as the sum of two terms (e.g. $\frac{5+2x}{2x}=\frac{5}{2x}+\frac{2x}{2x}=\frac{5}{2x}+1$). Students definitely demonstrated more understanding of the difference between terms and factors and dividing common factors in this problem set than previous ones.

Problem Set #8 (with answers)

In Column A, simplify if possible. *Show all steps.*
In Column B, substitute values for *a* and *b* to justify your response.
(Be sure your choice of *a* and *b* is part of the domain.)

A	B (for *a* = 4, *b* = 5)
$\frac{a+b}{a}$; cannot be simplified	$\frac{4+5}{4}=\frac{9}{4}$
$\frac{a+b}{ab}$; cannot be simplified	$\frac{4+5}{4\cdot 5}=\frac{9}{20}$
$\frac{3(a+b)}{3+a}$; cannot be simplified	$\frac{3(4+5)}{3+4}=\frac{27}{7}$
$\frac{2a\cdot 3b}{3a\cdot 2b}=\frac{6ab}{6ab}=1$	$\frac{2\cdot 4\cdot 3\cdot 5}{3\cdot 4\cdot 2\cdot 5}=\frac{120}{120}=1$
$\frac{3(a+b)}{3a+b}$; cannot be simplified	$\frac{3(4+5)}{3\cdot 4+5}=\frac{27}{17}$
$\frac{4a\cdot 8b}{4+8b}=\frac{4(8ab)}{4(1+2b)}=\frac{8ab}{1+2b}$	$\frac{4\cdot 4\cdot 8\cdot 5}{4+8\cdot 5}=\frac{640}{44}=\frac{4\cdot 160}{4\cdot 11}$

	$\frac{8 \cdot 4 \cdot 5}{1 + 2 \cdot 5} = \frac{160}{11}$
$\frac{3a(a+b)}{3a+b}$; cannot be simplified	$\frac{3 \cdot 4(4+5)}{3 \cdot 4 + 5} = \frac{108}{17}$

Each student should select his/her own numbers to substitute for *a* and *b*, and compare results with partners or group members. Some students may select numbers that are not mutually prime and sometimes allow the fraction to be simplified. Students should try to determine whether it is a "special case" or whether the expression can always be simplified for any values of *a* and *b*. These examples can also be used for a full class discussion to summarize the characteristics that allow rational expressions to be simplified.

Problem Set #9 (with answers)

In Column A, simplify if possible. *Show all steps.*
In Column B, substitute values for *a* to justify your response.
(Be sure your choice of *a* is part of the domain.)

A	B (for *a*=3)
$\frac{6a^2+a}{3a^2+a} = \frac{a(6a+1)}{a(3a+1)} = \frac{6a+1}{3a+1}$	$\frac{6 \cdot 3^2 + 3}{3 \cdot 3^2 + 3} = \frac{57}{30} = \frac{19 \cdot 3}{10 \cdot 3} = \frac{19}{10} = \frac{6 \cdot 3 + 1}{3 \cdot 3 + 1}$
$\frac{a^2-16}{a^2+8a+16} = \frac{(a-4)(a+4)}{(a+4)^2} = \frac{a-4}{a+4}$	$\frac{3^2-16}{3^2+8 \cdot 3+16} = \frac{-7}{49} = \frac{-1}{7} = \frac{3-4}{3+4}$
$\frac{a^2+a-6}{a^2-a-12} = \frac{(a+3)(a-2)}{(a+3)(a-4)} = \frac{a-2}{a-4}$	$\frac{3^2+3-6}{3^2-3-12} = \frac{6}{-6} = -1 = \frac{3-2}{3-4}$
$\frac{a^2-5a}{5a-a^2} = \frac{a^2-5a}{-(a^2-5a)} = -1$	$\frac{3^2-5 \cdot 3}{5 \cdot 3 - 3^2} = \frac{-6}{6} = -1$
$\frac{8a^2+6a-9}{16a^2-9} = \frac{(4a-3)(2a+3)}{(4a-3)(4a+3)} = \frac{2a+3}{4a+3}$	$\frac{8 \cdot 3^2 + 6 \cdot 3 - 9}{16 \cdot 3^2 - 9} = \frac{81}{135} = \frac{9}{15} = \frac{2 \cdot 3 + 3}{4 \cdot 3 + 3}$

These problems are traditional problems found in any algebra textbook. Students must factor the numerator and denominator completely to find common factors that can be divided. If they try to "cancel" *terms* from the numerator and denominator, students should find that they made an error when they substitute values for the variable and get different numerical results from the original expression.

Activity #3 – Operations with Rational Expressions Using Analogical Reasoning

Essential Question: How do we add, subtract, multiply and divide rational expressions? (CCSS.Math.Content.HSA.APR.D.7)

Once students have practiced factoring and simplifying rational expressions, they are ready to perform arithmetic operations (add, subtract, multiply, or divide) with them. This activity adds an analogical reasoning table to practice problems. Before beginning any operation, students must identify any values of the variable that are not part of the domain. Students will show their steps to perform the given operation in the first column of the table. Then, they will make a corresponding entry in the second column by substituting a value (checking that it is in the domain) into the equivalent expression at each step. Example 5 in the Background Content section demonstrates the process for subtraction. I provide examples of multiplication, division and addition here; the process/template can be used with any textbook problem sets.

Example: Multiplication

RATIONAL EXPRESSIONS **Domain: x ≠ -4**	**RATIONAL NUMBERS** **(for x = 3)**
$\frac{3x+12}{8}\cdot\frac{16x}{9x+36}$	$\frac{3\cdot 3+12}{8}\cdot\frac{16\cdot 3}{9\cdot 3+36}=\frac{21}{8}\cdot\frac{48}{63}=2$
$\frac{3(x+4)}{8}\cdot\frac{16x}{9(x+4)}$	$\frac{3(3+4)}{8}\cdot\frac{16\cdot 3}{9(3+4)}=\frac{21}{8}\cdot\frac{48}{63}=2$
$\frac{3\cdot 16x(x+4)}{8\cdot 9(x+4)}$	$\frac{3\cdot 16\cdot 3(3+4)}{8\cdot 9(3+4)}=\frac{3\cdot 7\cdot 3\cdot 16}{8\cdot 9\cdot 7}=\frac{21\cdot 48}{8\cdot 63}$
$\frac{3\cdot 16\cdot x}{8\cdot 9}$	$\frac{3\cdot 16\cdot 3}{8\cdot 9}=\frac{144}{72}=2$
$\frac{3\cdot 8\cdot 2\cdot x}{8\cdot 3\cdot 3}=\frac{2x}{3}$	$\frac{2\cdot 3}{3}=2$

Example: Division

RATIONAL EXPRESSIONS **Domain: x ≠ 3, -3, -4, 5**	**RATIONAL NUMBERS** **(for x = 2)**
$\frac{4x+12}{2x-10}\div\frac{x^2-9}{x^2-x-20}$	$\frac{4\cdot 2+12}{2\cdot 2-10}\div\frac{2^2-9}{2^2-2-20}=\frac{20}{-6}\div\frac{-5}{-18}$
$\frac{4x+12}{2x-10}\cdot\frac{x^2-x-20}{x^2-9}$	$\frac{4\cdot 2+12}{2\cdot 2-10}\cdot\frac{2^2-2-20}{2^2-9}=\frac{20}{-6}\cdot\frac{-18}{-5}$
$\frac{4(x+3)}{2(x-5)}\cdot\frac{(x-5)(x+4)}{(x+3)(x-3)}$	$\frac{4(2+3)}{2(2-5)}\cdot\frac{(2-5)(2+4)}{(2+3)(2-3)}=\frac{4\cdot 5}{2\cdot -3}\cdot\frac{-3\cdot 6}{5\cdot -1}$ $=\frac{20}{-6}\cdot\frac{-18}{-5}=\frac{-360}{30}-12$
$\frac{2(x+4)}{(x-3)}$	$\frac{2(2+4)}{(2-3)}=\frac{2\cdot 6}{-1}=-12$

Example: Addition

RATIONAL EXPRESSIONS Domain: x ≠ 1, 2, 3	RATIONAL NUMBERS (for x = 4)
$\frac{6}{x^2-5x+6}+\frac{3}{x^2-3x+2}$	$\frac{6}{4^2-5\cdot4+6}+\frac{3}{4^2-3\cdot4+2}=\frac{6}{2}+\frac{3}{6}$
$\frac{6}{(x-2)(x-3)}+\frac{3}{(x-1)(x-2)}$	$\frac{6}{(4-2)(4-3)}+\frac{3}{(4-1)(4-2)}$ $=\frac{6}{2\cdot1}+\frac{3}{3\cdot2}=\frac{6}{2}+\frac{3}{6}=\frac{7}{2}$
$\frac{6}{(x-2)(x-3)}\cdot\frac{(x-1)}{(x-1)}$ $+\frac{3}{(x-1)(x-2)}\cdot\frac{(x-3)}{(x-3)}$	$\frac{6}{(4-2)(4-3)}\cdot\frac{(4-1)}{(4-1)}+$ $\frac{3}{(4-1)(4-2)}\cdot\frac{(4-3)}{(4-3)}=\frac{6\cdot3}{2\cdot3}+\frac{3\cdot1}{6\cdot1}$
$\frac{6(x-1)}{(x-1)(x-2)(x-3)}$ $+\frac{3(x-3)}{(x-1)(x-2)(x-3)}$	$\frac{6(4-1)}{(4-1)(4-2)(4-3)}$ $+\frac{3(4-3)}{(4-1)(4-2)(4-3)}=$ $\frac{6\cdot3}{3\cdot2\cdot1}+\frac{3\cdot1}{3\cdot2\cdot1}=\frac{18}{6}+\frac{3}{6}=\frac{21}{6}=\frac{7}{2}$
$\frac{6x-6+3x-9}{(x-1)(x-2)(x-3)}$	$\frac{6\cdot4-6+3\cdot4-9}{(4-1)(4-2)(4-3)}=\frac{21}{6}=\frac{7}{2}$
$\frac{9x-15}{(x-1)(x-2)(x-3)}$ $=\frac{3(3x-5)}{(x-1)(x-2)(x-3)}$	$\frac{9\cdot4-15}{(4-1)(4-2)(4-3)}=\frac{21}{6}=\frac{7}{2}$ $\frac{3(3\cdot4-5)}{(4-1)(4-2)(4-3)}=\frac{3\cdot7}{6}=\frac{21}{6}=\frac{7}{2}$

While I predict resistance from students about doing “extra” work for each problem, I believe the end result of deeper understanding and being able to construct strong arguments for or against statements is worth the pain of listening to their complaints.

Common Core State Standards Addressed

CCSS.Math.Content.HSA.SSE.A.1.a: Interpret parts of an expression, such as terms, factors, and coefficients.

CCSS.Math.Content.HSA.APR.D.7: Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

CCSS.Math.Content.HSA.REI.A.1: Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

CCSS.Math.Content.HSA.REI.A.2: Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

CCSS.Math.Practice.MP2: Reason abstractly and quantitatively.

CCSS.Math.Practice.MP3: Construct viable arguments and critique the reasoning of others.

CCSS.Math.Practice.MP7: Look for and make use of structure.

Resources for Classroom Activities Related to Rational Functions

Davis, Anna A., Ronald E. Zielke, Jessica L. Lickeri, Ruth Dover, and Corey Andreasen. "Rational Functions: A New Perspective." *The Mathematics Teacher* 104, no. 7 (2011): 538-49. Accessed August 20, 2014. www.jstor.org/stable/20876942.
Activity using perspective drawings and data collection to derive rational functions.

Denson, Philinda. "Algebraic Fractions Messages." *The Mathematics Teacher* 82, no. 2 (1989): 128. Accessed July 13, 2015. www.jstor.org/stable/27966144.
Activity that requires students to manipulate rational expressions to find a message.

Hodgson, Theodore, and David Schultz. "A Refreshingly Rational Approach to the Center of Mass." *Mathematics Teacher* 108, no. 9 (2015): 710-15.
Students model center of mass of soda in a can from full to empty.

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Blubaugh, William L. "Why Cancel?" *The Mathematics Teacher* 81, no. 4 (1988): 300-02. Accessed July 14, 2015. www.jstor.org/stable/27965800.

Brown, George, and Robert Quinn. "Algebra Students' Difficulty with Fractions: An Error Analysis." *The Australian Mathematics Teacher*, 2006, 28-40.

Darley, Joy W. "Traveling from Arithmetic to Algebra." *Mathematics Teaching in the Middle School* 14, no. 8 (2009): 458-64. Accessed July 23, 2015. www.jstor.org/stable/41182728.

Darley, Joy W., and Barbara B. Leapard. "Connecting Arithmetic to Algebra." *Teaching Children Mathematics* 17, no. 3 (2010): 184-91. Accessed June 28, 2015. www.jstor.org/stable/41199629.

Eisen, Arron. "A New Math Term Is Born: FFOO." *The Mathematics Teacher* 92, no. 5 (1999): 380. Accessed July 23, 2015. www.jstor.org/stable/27971012.
The term FFOO is a silly way to remind students that they can write the number one an infinite number of ways to find equivalent expressions.

Gentner, Dedre. "Psychology of analogical reasoning," in *Encyclopedia of Cognitive Science*, edited by L. Nadel, 106-112. Nature Publishing Group, 2003.

Grossman, August. "An Analysis of the Teaching of Cancellation in Algebraic Fractions." *The Mathematics Teacher* 17, no. 2 (1924): 104-09. Accessed July 13, 2015. www.jstor.org/stable/27950589.

Hadley, Hazel. "Schema for Operating with Algebraic Fractions." *The Mathematics Teacher* 79, no. 4 (1986): 258-60. Accessed July 13, 2015. www.jstor.org/stable/27964886.
Provides a flow chart for working with rational expressions, a very procedural approach.

Hornsby, Jr., E. John. "Composing "Interesting" Exercises Involving Rational Expressions." *The Mathematics Teacher* 77, no. 3 (1984): 216-19. Accessed July 23, 2015. www.jstor.org/stable/27963970.

Hornsby, Jr., E. John, and Jeffrey A. Cole. "Rational Functions: Ignored Too Long in the High School Curriculum." *The Mathematics Teacher* 79, no. 9 (1986): 691-98. Accessed June 30, 2014. www.jstor.org/stable/27965164.
Author explains how to use number sense to help graph rational functions.

Martinez, Joseph G. R. "Helping Students Understand Factors and Terms." *The Mathematics Teacher* 81, no. 9 (1988): 747-51. Accessed July 23, 2015. www.jstor.org/stable/27966036.

Orzechowski, Larry. "Rational Expressions and Rational Equations: Consistency Versus Simplicity." *The Mathematics Teacher* 78, no. 9 (1985): 682-84. Accessed

July 23, 2015. www.jstor.org/stable/27964712.
Author recommends solving rational equations by first combing rational expressions with common denominators.

Storer, W. O., and Board of Studies for Mathematics. "An Analysis of Errors Appearing in a Test on Algebraic Fractions." *The Mathematical Gazette*, 1956, 24-33.

Szymanski, Ted. "A Guide to Shotgun Canceling." *Research and Teaching in Developmental Education* 16, no. 1 (1999): 113-16. Accessed July 14, 2015. www.jstor.org/stable/42802058.

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[2] William Blubaugh, "Why Cancel?" *The Mathematics Teacher* 81 (1988): 301, accessed July 14, 2015, www.jstor.org/stable/27965800.
[3] Ted Szymanski, "A Guide To Shotgun Canceling," *Research and Teaching in Developmental Education* 16 (1999): 113, accessed July 14, 2015, www.jstor.org/stable/42802058.
[4] Joy W. Darley, "Traveling from Arithmetic to Algebra," *Math Teaching in the Middle School* 14 (2009): 460, accessed July 23, 2015, www.jstor.org/stable/41182728.
[5] Joy W. Darley and Barbara B. Leapard, "Connecting Arithmetic to Algebra," *Teaching Children Mathematics* 17 (2010): 185, accessed June 28, 2015, www.jstor.org/stable/41199629.
[6] Paul Bertha, "Analogy and Analogical Reasoning," *The Stanford Encyclopedia of Philosophy* (2013), accessed October 11, 2015. http://plato.stanford.edu/archives/fall2013/entries/reasoning-analogy/.
[7] Dedre Gentner, "Psychology of Analogical Reasoning," in *Encyclopedia of Cognitive Science*, ed. L. Nadel (Nature Publishing Group, 2003), 109.
[8] Bartha, "Analogical Reasoning."
[9] Bartha, "Analogical Reasoning."
[10] Joseph Martinez, "Helping Students Understand Factors and Terms," *Mathematics Teacher* 81 (1988): 747-51, accessed July 23, 2015, www.jstor.org/stable/27966036.
[11] Martinez, "Factors and Terms," 751.
[12] Arron Eisen, "A New Math Term is Born: FFOO," *The Mathematics Teacher* 92 (1999): 380, accessed July 23, 2015, www.jstor.org/stable/27971012.
[13] Larry Orzechowski, "Rational Expressions and Rational Equations: Consistency Versus simplicity," *The Mathematics Teacher* 78 (1985): 682-684 accessed July 23, 2015, www.jstor.org/stable/27964712.

Curriculum Unit Title: Meaningful Operations with Rational Expressions through Analogical Reasoning

Author: Nancy Rudolph

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.
Each step in solving a simple equation follows from the equality of numbers asserted at the previous step, and viable arguments to justify the solution method can be constructed.

ESSENTIAL QUESTION(S) for the UNIT

How can the properties of the real number system be useful when working with polynomials and rational expressions?
In what ways can the problem be solved, and why should one method be chosen over another?

CONCEPT A	CONCEPT B	CONCEPT C
Factors versus Terms	Simplifying Rational Numbers and Expressions	Operations with Rational Expressions Using Analogical Reasoning

ESSENTIAL QUESTIONS A	ESSENTIAL QUESTIONS B	ESSENTIAL QUESTIONS C
How do we interpret parts of an expression, such as terms, factors, and coefficients?	In what ways are rational expressions analogous to rational numbers?	How do we add, subtract, multiply and divide rational expressions?

VOCABULARY A	VOCABULARY A	VOCABULARY A
factors terms coefficients variables expression polynomials	rational number rational expression analogous simplify equivalent expressions multiplicative identity closure property	factoring arithmetic operations

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

Examples of a template for students to justify their steps in simplifying and/or operating on rational expressions using Analogical Reasoning are provided within the unit.